CHAIN TRANSITIVITY IN HYPERSPACES

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Abstract. Given a non-empty compact metric space X and a continuous function $f\colon X \to X$, we study the dynamics of the induced maps on the hyperspace of non-empty compact subsets of X and on various other invariant subspaces thereof, in particular symmetric products. We show how some important dynamical properties transfer across induced systems. These amongst others include, chain transitivity, chain (weakly) mixing, chain recurrence, exactness by chains. From our main theorem we derive an ε-chain version of Furstenberg's celebrated 2 implies n Theorem. We also show the implications our results have for dynamics on continua.

1. Introduction

Given a non-empty compact metric space X, the study of the dynamics given by a continuous map $f\colon X \to X$ can be approached from many different angles. The most direct strategy, by studying individual orbits, can indeed be very fruitful, and most of the topological dynamics has built upon this premise.

In reality, however, one rarely knows precisely the state of a physical system. The observables usually come with some uncertainty and it therefore makes sense to study probability distributions on X and how they evolve. As Bauer et al. nicely put it in their paper [3] from 1975:

> "The elements of $M(K)$ can be viewed as statistical states, representing imperfect knowledge of the system. The elements of X are imbedded in $M(X)$ as the pure states."

To a topologist, it seems natural to dismiss probabilities and study how the support of a probability distribution evolve over time, i.e. to study the induced dynamics on 2^X, the hyperspace of compact subsets of X. In fact, Bauer et al. initiate this line of research in that very paper. This made even more sense with the development of computers, as in order to keep track of the precision, one usually computes the orbit of an interval representing lower and upper bounds for the initial state. It has been argued [6] that, from a computational and domain theoretic point of view, this is the natural approach to dynamical systems.

Related to this is also the fact that we are studying chain versions of well-known topological properties. Recall that a finite precision numerical simulation of some dynamics inevitably produces ε-pseudo-orbits, where ε is determined by the precision of the machine. It is therefore natural to study, say, chain transitivity, as a chain transitive system, when simulated on a finite precision machine, may well

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exhibit the same behaviour as a truly transitive system. What is worse, this cannot be improved by increasing the precision of the machine, as long as it is kept finite.

In this paper we show that for a chain transitive map f either majority of the induced dynamics on the hyperspaces are chain transitive, or none of them is. This is the content of Theorem 3 ((A) and (D) parts) and witnesses that the hyperspaces under consideration, symmetric and Cartesian products, are rigidly embedded within 2^X. Recall that Furstenberg's celebrated 2 implies n Theorem, see [7, Proposition II.3], establishes an analogous result for the classic version of transitivity and the implication (D1) $\Longrightarrow$ (D2) in our Theorem can thus be seen as an ε-chain version of Furstenberg's result. This is extracted in Corollary 4.

Parts (E) and (F) establish the equivalence of chain transitivity of the hyperspaces to other properties of interest, chain (weakly) mixing, exactness by chains, total chain transitivity. This is in contrast to the classic versions of these properties as those are distinguishable (see Examples 1 and 2).

Property (C) bears resemblance to Šarkovs′kiĭ's weak incompressibility which, as was demonstrated by Barwell et al. in [2], has everything to do with chain transitivity, hence this connection should not take us by surprise. Finally, part (B) characterises chain transitivity in hyperspaces by imposing a combinatorial restriction on pseudo-orbits.

Not all results presented here are completely new. In [22] Yang proves the sequence of equivalences (B1) $\iff$ (E1) $\iff$ (F1) $\iff$ (F2). We were informed about this after independently proving our result. Since Yang's proof appears only in Chinese language, we decided to include our own proof. In [19] Richeson et al. prove equivalence (F1) $\iff$ (F2), and they also show that these are equivalent to chain transitivity and chain recurrence of f for the case of continua. In [11] Khan et al. prove that chain transitivity and chain recurrence of 2^f implies the same properties for f, that (F2) is equivalent to 2^f being chain mixing, and that for continua all the implications go in both directions. This being said, to the best of our knowledge, this is the first comprehensive account of the topic of chain transitivity in hyperspaces.

We do not however obtain a satisfying characterisation of chain transitivity for the hyperspace of subcontinua of X, and a recent paper [16] by Matviichuk et al. is a step forward in this direction for the case of interval maps.

The related work of other authors investigating different aspects of dynamics in hyperspaces includes: the previously mentioned paper [3] where Bauer et al. prove that f is weakly/mild/strong mixing if and only if the induced map on 2^X is weakly/mild/strong mixing respectively; a characterization of the transitivity of the induced map on 2^X in [4] by Banks, in [18] by Peris, and in [1] by Acosta et al.; a study of entropy by Kwietniak et al. in [13]; shadowing property by Wu et al. in [21]; periodicity, recurrence, quasi-periodicity, (non-)wandering points, shadowing, exactness in hyperspaces by Gómez et al. in [8]; and very recently disjointness by Li et al. in [14].

The rest of this paper is organised as follows. In Section 2 we introduce our notation, the main theorem is stated in Section 3, and its proof occupies Section 4. The application to the continua is left for Section 5 where we also establish another interesting dichotomy, namely Theorem 15 stating that a space admitting weakly chain mixing must be either connected, or have infinitely many components.

2. Preliminaries

Throughout the paper, unless stated otherwise, X denotes a non-empty compact metric space and $f\colon X \to X$ a continuous map on it. We usually fix some metric

d on X to work with. This however is not a restriction as most of the interesting dynamical properties are of topological nature and hence independent of the choice of the metric. By

$$2^X = \{A \subseteq X \colon A \text{ is non-empty and closed}\}$$

we denote the *hyperspace of closed non-empty subsets of* X. For any $r > 0$ and any $A \in 2^X$, the *open ball about* A *of radius* r is given by

$$N_X(A, r) = \{x \in X \colon d(x, A) < r\}.$$

For the special case when $A = \{x\}$ we will denote this by $N_X(x, r)$ instead of $N_X(\{x\}, r)$. The *closure in* X of a subset $A \subseteq X$ is denoted by $Cl_X(A)$. If it is clear which space we are considering, we just denote it as $Cl(A)$.

The space 2^X comes equipped with a natural metric $H \colon 2^X \times 2^X \to [0, \infty)$ given by

$$H(A, B) = \inf\{\varepsilon > 0 \colon A \subseteq N_X(B, \varepsilon) \text{ and } B \subseteq N_X(A, \varepsilon)\},$$

which is often called the *Hausdorff metric*. It turns out that the topology on 2^X induced by this metric is compact and coincides with the abstractly defined *Vietoris' topology* given by the basis

$$\mathcal{B} = \{\langle U_1, U_2, \dots, U_m\rangle \ : \ U_i \text{ is open for each } i \in \{1, 2, ..., m\}, m \in \mathbb{N}\},$$

where

$$\langle U_1, U_2, ..., U_m\rangle = \left\{A \in 2^X \colon A \subseteq \bigcup_{i=1}^{m} U_i \text{ and } A \cap U_i \neq \emptyset, i \in \{1, 2, ..., m\}\right\}.$$

For the proof of these and related results see [15] and [17, Theorem 0.11 and 0.13].

A natural way to define the induced map $2^f : 2^X \to 2^X$ is by the formula

$$2^f(A) = f(A) = \{f(x) : x \in A\}, \text{ for } A \in 2^X.$$

It is immediate that all of the following

- $C(X) = \{A \in 2^X \colon A \text{ is connected}\}$ — the *hyperspace of subcontinua of* X,
- $C_n(X) = \{A \in 2^X \colon A \text{ has at most } n \text{ components}\}$,
- $F_n(X) = \{A \in 2^X \colon A \text{ has at most } n \text{ points}\}$ — the *n-fold symmetric product of* X,
- $F(X) = \bigcup_{n=1}^{\infty} F_n(X)$ — the collection of all finite subsets of X

are 2^f-invariant closed subspaces of 2^X, and it is natural to study restrictions (in both the domain and the co-domain) of 2^f to these subspaces for which we introduce the following notation

- $2^f|_{C(X)} = C(f)$,
- $2^f|_{C_n(X)} = C_n(f)$,
- $2^f|_{F_n(X)} = f_n$,
- $2^f|_{F(X)} = f^{<\omega}$.

Occasionally we will write just f for any of the above maps as this does not lead to any confusion, and is useful to keep the notation simple, especially when we need to refer to the k^{th} iterate of the map 2^f which we simply denote by f^k.

Remark 1. When using the Vietoris' topology in a symmetric product, we will denote by $\langle U_1, U_2, ..., U_m\rangle_n$ the intersection of a basic open set $\langle U_1, U_2, ..., U_m\rangle$ with $F_n(X)$

$$\langle U_1, U_2, ..., U_m\rangle_n = \langle U_1, U_2, ..., U_m\rangle \cap F_n(X).$$

The usual n-fold Cartesian product will also be of interest as the n-fold symmetric product can be seen as a quotient of this space. Somewhat unconventionally we denote the product space $\underbrace{X \times X \times \cdots \times X}_{n-times}$ by $X^{(n)}$ and the induced map by $f^{(n)}$. This is not to be confused with f^n which is simply the n^{th} iterate of f.

3. Main theorem

In order to state our main result we need to recall the definitions of chain versions of some well-studied dynamical properties.

Given a $\delta > 0$, a *δ-pseudo orbit* is a finite or infinite sequence of points $\langle x_0, x_1, \ldots \rangle$ such that $d(f(x_i), x_{i+1}) < \delta$. If we have a finite δ-pseudo orbit: $\langle x_0, x_1, \ldots, x_n \rangle$, we will call it *$\delta$-chain* and we say that it has *length n*. A subset Λ of X is *internally chain transitive* (or alternatively *f is internally chain transitive on* Λ) if for every pair of points x, y in Λ and every $\delta > 0$ there is a δ-chain $\langle x = x_0, x_1, \ldots, x_m = y \rangle \subseteq \Lambda$ between x and y. In the special case when $\Lambda = X$, we say that f (or X) is *chain transitive* (CT).

If $a, b, c \in X$ and Γ_1, Γ_2 are two δ-chains in X, $\Gamma_1 = \left\langle a = l_0^1, l_1^1, \ldots, l_{k_1}^1 = b \right\rangle$ and $\Gamma_2 = \left\langle b = l_0^2, l_1^2, \ldots, l_{k_2}^2 = c \right\rangle$, respectively, then $\Gamma_1 + \Gamma_2$ denotes the *concatenation* of Γ_1 with Γ_2,

$$\Gamma_1 + \Gamma_2 = \left\langle a = l_0^1, l_1^1, \ldots, l_{k_1}^1 = b = l_0^2, l_1^2, \ldots, l_{k_2}^2 = c \right\rangle.$$

Note that the length of $\Gamma_1 + \Gamma_2$ is the sum of lengths of Γ_1 and Γ_2. If Γ is a δ-chain with the same starting and ending point, $\Gamma = \langle a = l_0, l_1, \ldots, l_k = a \rangle$, and $m \geq 1$, $m\Gamma$ denotes $\underbrace{\Gamma + \Gamma + \cdots + \Gamma}_{m-times}$.

Definition 1. We say that f is *chain weakly mixing* (or *weakly mixing by chains*) if the function $f^{(2)} \colon X^{(2)} \to X^{(2)}$ is chain transitive; f is *totally chain transitive* (or *totally transitive by chains*) if for every $n \geq 1$, the function $f^n \colon X \to X$ is chain transitive; f is *exact by chains* if for every $\varepsilon > 0$ and every non-empty open subset U of X, there is a positive integer $n_\varepsilon \geq 1$ such that for every $x \in X$ there exists $u \in U$ and an ε-chain $\langle u = a_0, a_1, \ldots, a_{n_\varepsilon} = x \rangle$ from u to x with length exactly n_ε.

Recall that a map $f \colon X \to X$ is called *exact* if for every non-empty open set $U \subseteq X$, there exists a positive integer m such that $f^m(U) = X$, and the map f is called *weakly mixing* if $f \times f \colon X \times X \to X \times X$ is transitive. It is known that every exact function is weakly mixing ([18, Theorem 2.1] and [9, Lemma 5]), and every weakly mixing function is totally transitive ([4, Theorem 1]). It turns out that the chain versions of these properties are in fact equivalent (see Theorem 3). This is not true for the classic notions as the two examples below show.

Example 1. Let $f_\theta \colon S^1 \to S^1$ be an irrational rotation on the unit circle. Then f_θ is totally transitive. Let U and V be two sufficiently small open arcs, diametrically opposed, then for every n, $f_\theta^n(U) \cap f_\theta^n(V) = \emptyset$, which implies that $[(f_\theta \times f_\theta)^n(U \times V)] \cap (U \times U) = \emptyset$. Therefore, f_θ is not weakly mixing.

Example 2. Let $I = [0, 1]$ be the unit interval and let $T \colon I \to I$ be the tent map given by

$$T(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}] \\ 2 - 2x & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Let $X_\infty = \varprojlim\{T, I\}$ and let $\sigma_T \colon X_\infty \to X_\infty$ be the map given by

$$\sigma_T(x_1, x_2, \ldots) = (T(x_1), x_1, x_2, \ldots).$$

Since T is exact, then T is weakly mixing. By [5, Theorem 7], σ_T is weakly mixing but it can not be exact because it is an homeomorphism.

Definition 2. We say that f is *chain mixing* (or *mixing by chains*) if for every $\varepsilon > 0$ there exists a positive integer N such that for all $n \geq N$ and for any pair $x, y \in X$, there exists an ε-chain from x to y of length n, $\langle x = x_0, x_1, x_2, \ldots, x_n = y\rangle$; f is *chain recurrent* if for every $x \in X$ and every $\varepsilon > 0$, there is an ε-chain from x to itself.

Theorem 3 (Main theorem)**.** *Let X be a compact metric space and let $f\colon X \to X$ be a continuous function. Then the following are equivalent:*

(A1) *$f^{<\omega}$ is CT,*
(A2) *2^f is CT,*
(A3) *f_n is CT for some $n \geq 2$,*
(A4) *f_n is CT for any $n \geq 1$,*

(B1) *f is CT and there exist $z \in X$ such that for any $\delta > 0$ there are two δ-chains from z to itself with co-prime lengths,*
(B2) *f is CT and for any $z \in X$ and any $\delta > 0$ there are two δ-chains from z to itself with co-prime lengths,*

(C) *f is CT and for every pair of non-empty disjoint open proper subsets of X, U and V, we have either $Cl_X(f(U)) \nsubseteq V$ or $Cl_X(f(V)) \nsubseteq U$,*

(D1) *$f^{(n)}$ is CT for some $n \geq 2$,*
(D2) *$f^{(n)}$ is CT for any $n \geq 1$,*

(E1) *f is chain weakly mixing,*
(E2) *f is exact by chains,*
(E3) *for every $z \in X$ and each $\varepsilon > 0$, there is a positive integer $n_\varepsilon \geq 1$ such that for each $x \in X \setminus \{z\}$, there is an ε-chain of length n_ε from z to x,*

(F1) *f is totally chain transitive,*
(F2) *f is chain mixing.*

Furthermore, if any, and hence all of the properties above hold then f is CT.

The implication (D1) $\implies$ (D2) in the previous theorem can be interpreted as a chain version of the Theorem of Furstenberg [7, Proposition II.3].

Corollary 4. *If f is chain weakly mixing, then $f^{(n)}\colon X^{(n)} \to X^{(n)}$ is chain transitive for every $n \geq 1$.*

We also get

Corollary 5. *If a chain transitive map f has a fixed point then all of the statements in Theorem 3 hold.*

This is not a necessary condition however. One can easily see this by considering an irrational rotation of the circle which clearly is chain transitive, and hence by Corollary 13 below satisfies all the other conditions in Theorem 3, but has no periodic points whatsoever.

4. The proof of the main theorem

Lemma 6 ([12], Proposition 2.6)**.** *Let X be a compact metric space and let $f\colon X \to X$ be a continuous function. If f is chain transitive, then f is onto.*

Proof. Let $y \in X$ and fix another point $x \in X$. Since f is chain transitive, then for every positive integer j, there is a $\frac{1}{j}$-chain from x to y. This implies that there exists a sequence $\{x^j_{n(j)-1}\}_{j=1}^{\infty}$ such that $d\left(f\left(x^j_{n(j)-1}\right), y\right) < \frac{1}{j}$. Since X is compact, we may assume that for some $x_0 \in X$, $x^j_{n(j)-1} \to x_0$ as $j \to \infty$. Since f is continuous $f(x_0) = y$. Thus, f is onto. $\square$

The following proposition proves the last claim of Theorem 3. It actually shows a bit more, and is important in its own right. Example 3 below it shows that the converse to the proposition does not hold in general. A partial converse however, in case X is continuum, holds and will be given in Theorem 13.

Proposition 7. *Let X be a compact metric space and let $f\colon X \to X$ be a continuous function. If $C(f)$, $C_n(f)$, f_n, $f^{<\omega}$ or 2^f, for any n, is chain transitive, then f is chain transitive.*

Proof. Let $\delta > 0$, let $\bar{f}\colon \mathcal{X} \to \mathcal{X}$ be the induced map, where $\mathcal{X} \in \{C(X), C_n(X), F_n(X), F(X), 2^X\}$, and assume that $\bar{f}$ is chain transitive. Let $x, y \in X$. Since $\{x\}, \{y\} \in \mathcal{X}$, there is a δ-chain in $\mathcal{X}$, $\langle \{x\} = A_0, A_1, A_2, \ldots, A_k = \{y\}\rangle$ with $k \geq 1$. Now, since $H(\bar{f}(\{x\}), A_1) < \delta$, $\{f(x)\} \subseteq N_X(A_1, \delta)$, thus, there is $a_1 \in A_1$ such that $d(f(x), a_1) < \delta$. Since $H(\bar{f}(A_1), A_2) < \delta$, then $\bar{f}(A_1) \subseteq N_X(A_2, \delta)$, there is $a_2 \in A_2$ for which $d(f(a_1), a_2) < \delta$. Following this process there is $a_{i+1} \in A_{i+1}$ such that $d(f(a_i), a_{i+1}) < \delta$. Thus, the sequence $\langle x = a_0, a_1, a_2, \ldots, a_k = y\rangle$ is a δ-chain from x to y in X. $\square$

Example 3. Let $X = \{a, b\}$ and let $f\colon X \to X$ given by $f(a) = b$ and $f(b) = a$. It is clear that f is transitive, which implies that f is chain transitive. Nevertheless, 2^f is not chain transitive. If we take $\delta = \frac{d(a,b)}{2}$, there is no a δ-chain from $\{a\}$ to $\{a, b\}$.

4.1. **Equivalence of (A1) – (A4).** We first prove an auxiliary lemma.

Lemma 8. *Let X be a compact metric space, let $f\colon X \to X$ be a continuous function and let Y be a dense and invariant subset of X. Then f is chain transitive if and only if $f|_Y$ is chain transitive.*

Proof. Assume that f is chain transitive, let $a, b \in Y$ and let $\varepsilon > 0$. Since X is compact, there is $\delta > 0$ such that $0 < \delta < \frac{\varepsilon}{2}$ and if $d(s, t) < \delta$, then $d(f(s), f(t)) < \frac{\varepsilon}{2}$ for every $s, t \in X$. By hypothesis, there is a $\frac{\delta}{2}$-chain in X from a to b, $\langle a = z_0, z_1, z_2, \ldots, z_k = b\rangle$, with $k \geq 1$. Since Y is dense in X, we have that, for each $i \in \{1, 2, \ldots, k-1\}$, there is $t_{i+1} \in N_X\left(z_{i+1}, \frac{\delta}{2}\right) \cap Y$. Thus,

$$d(f(t_i), t_{i+1}) \leq d(f(t_i), f(z_i)) + d(f(z_i), z_{i+1}) + d(z_{i+1}, t_{i+1}) < \frac{\varepsilon}{2} + \frac{\delta}{2} + \frac{\delta}{2} = \frac{\varepsilon}{2} + \delta < \varepsilon.$$

Therefore, the sequence $\langle a, t_1, t_2, \ldots, t_{k-1}, b\rangle$ is an ε-chain in Y.

Now assume that $f|_Y$ is chain transitive, let $x, y \in X$, let $y' \in f^{-1}(y)$ and let $\varepsilon > 0$. Also, let $\delta > 0$ such that if $d(a, b) < \delta$, then $d(f(a), f(b)) < \varepsilon$. Since Y is dense in X, there are $z_0, z_1 \in Y$ such that $d(z_0, f(x)) < \varepsilon$ and $d(z_1, y') < \delta$. This implies that $d(f(z_1), y) < \varepsilon$. By hypothesis, there is an ε-chain in Y, $\langle z_0 = a_0, a_1, a_2, \ldots, a_k = z_1\rangle$, with $k \geq 1$. Thus, the sequence $\langle x, z_0 = a_0, a_1, \ldots, a_k = z_1, y\rangle$ is an ε-chain from x to y in X. $\square$

Recall that for every $A \in 2^X$ and any $\varepsilon > 0$, there is a finite set $K \in F(X) \subseteq 2^X$ such that $H(A, K) < \varepsilon$. This along with Lemma 8 gives (A1) $\iff$ (A2). The implication (A4) $\implies$ (A1) is also immediate as for two sets $A, B \in F(X)$ one can always find $n \in \mathbb{N}$ such that both A and B lie in $F_n(X)$. The chain in between provided by f_n also works as a chain in $F(X)$.

We now prove (A2) $\implies$ (A3), more precisely that (A2) implies chain transitivity for f_2. Assume that 2^f is chain transitive. Let $A = \{x_0, y_0\}$, $B = \{x, y\}$ be two points in $F_2(X)$, let $z \in X$, and let $\delta > 0$. Since 2^f is chain transitive, then there is a δ-chain $\Gamma_1 = \langle A = A_0, A_1, A_2, \ldots, A_n = \{z\}\rangle$ from A to

$\{z\}$ in 2^X. Since $H\left(f(A_i), A_{i+1}\right) < \delta$, there are points $x_i, y_i \in A_i$ such that $d\left(f(x_i), x_{i+1}\right) < \delta$ and $d\left(f(y_i), y_{i+1}\right) < \delta$, for every $i \in \{0, 1, 2, \ldots, n-1\}$. Let $A_i^* = \{x_i, y_i\}$. Then $\Gamma_1^* = \langle A^* = A_0^*, A_1^*, A_2^*, \ldots, A_n^* = \{z\}\rangle$ is a δ-chain from A to $\{z\}$ in $F_2(X)$. Now, since 2^f is chain transitive, then there is a δ-chain $\Gamma_2 = \langle \{z\} = B_0, B_1, B_2, \ldots, B_l = B\rangle$ from $\{z\}$ to B. Let us rename the points in $B = \{x, y\}$ to $B = \{x_l, y_l\}$. Since $H\left(f(B_{l-1}), B_l\right) < \delta$, then for $x_l \in B_l$, there is $x_{l-1} \in B_{l-1}$ such that $d\left(f(x_{l-1}), x_l\right) < \delta$, and for $y_l \in B_l$, there is $y_{l-1} \in B_{l-1}$ such that $d\left(f(y_{l-1}), y_l\right) < \delta$. Continuing this process we obtain the sets $B_i^* = \{x_i, y_i\}$ such that $d\left(f(x_{i-1}), x_i\right) < \delta$ and $d\left(f(y_{i-1}), y_i\right) < \delta$ for every $i \in \{1, 2, \ldots, l\}$. Thus, $\Gamma_2^* = \langle B_0^* = \{z\}, B_1^*, B_2^*, \ldots, B_l^* = B\rangle$ is a δ-chain from $\{z\}$ to B in $F_2(X)$. Therefore, the concatenation of $\Gamma_1^* + \Gamma_2^*$ is a δ-chain from A to B in $F_2(X)$.

To close the circle of equivalences (A1)$-$(A4) it remains to prove (A3) $\implies$ (A4). First note that the reasoning above, deriving chain transitivity of f_2 from (A2), mutatis mutandis, proves that (A3) also implies chain transitivity of f_2. It will therefore suffice to show that given a fixed $n \geq 2$, chain transitivity of f_n implies chain transitivity of f_{n+1}. To this end let A and B be points in $F_{n+1}(X)$ and let $\delta > 0$. Without lost of generality we may assume that $A = \{a_1, a_2, \ldots, a_{n+1}\}$ and $B = \{b_1, b_2, \ldots, b_{n+1}\}$. Let $A' = \{a_1, a_2, \ldots, a_n\}$, $B' = \{b_1, b_2, \ldots, b_n\}$. $A'' = \{a_1, a_2, \ldots, a_{n-1}\}$, $B'' = \{b_1, b_2, \ldots, b_{n-1}\}$. Since f_n is chain transitive, there are δ-chains:

$$\langle A' = A_0, A_1, A_2, \ldots, A_r = A''\rangle \text{ of length } r$$

and

$$\langle B'' = B_0, B_1, B_2, \ldots, B_t = B'\rangle \text{ of length } t$$

Then

$$\Gamma_1 = \left\langle A_0 \cup \{a_{n+1}\}, A_1 \cup \{f(a_{n+1})\}, A_2 \cup \{f^2(a_{n+1})\}, \ldots, A_r \cup \{f^r(a_{n+1})\}\right\rangle$$

is a δ-chain from A to $A'' \cup \{f^r(a_{n+1})\}$. Let $w \in f^{-t}(b_{n+1})$, then

$$\Gamma_2 = \left\langle B_0 \cup \{w\}, B_1 \cup \{f(w)\}, B_2 \cup \{f^2(w)\}, \ldots, B_t \cup \{f^t(w)\}\right\rangle$$

is a δ-chain from $B'' \cup \{w\}$ to B. Since $f_n(X)$ is chain transitive, then there is a δ-chain Γ_3 form $A'' \cup \{f^r(a_{n+1})\}$ to $B'' \cup \{w\}$. Thus, the concatenation $\Gamma_1 + \Gamma_3 + \Gamma_2$ is a δ-chain from A to B in $F_{n+1}(X)$.

4.2. **Equivalence of (B1) – (B2).** We now proceed to show (A4) $\implies$ (B2). We assume, in particular, that f_2 is chain transitive. Let $z \in X$ and $\delta > 0$ be arbitrary. Since f_2 is chain transitive then there exists a δ-chain from $\{z, f(z)\}$ to $\{z\}$ in $F_2(X)$. If this δ-chain has length r, then this implies that there are two different δ-chains from z to z of lengths r and $r+1$ as desired. Furthermore, by Proposition 7 f is CT.

Clearly (B2) implies (B1) and hence it will suffice to show that (B1) implies, say, (A3). To that end let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ be two points in $F_2(X)$ and let $\delta > 0$. In order to show chain transitivity of f_2, we will construct a δ-chain from A to B in $F_2(X)$. For $i \in \{1, 2\}$ let α_i be the shortest δ-chain from a_i to z, let β_i be the shortest δ-chain from z to b_i, and let γ_i be a δ-chain from z to z of length p_i. Also, let us assume that the length of α_i is k_i and the length of β_i is m_i. In order to have a δ-chain from A to B in $F_2(X)$, we need to find positive numbers r and t satisfying: $k_1 + r \cdot p_1 + m_1 = k_2 + t \cdot p_2 + m_2$. Without loss of generality, we may assume that $k_1 + m_1 > k_2 + m_2$. Thus, the numbers r and t should satisfy: $k_1 + m_1 - (k_2 + m_2) = t \cdot p_2 - r \cdot p_1$. Since $(p_1, p_2) = 1$, then it is always possible to find the numbers r and t (see Figure 1).

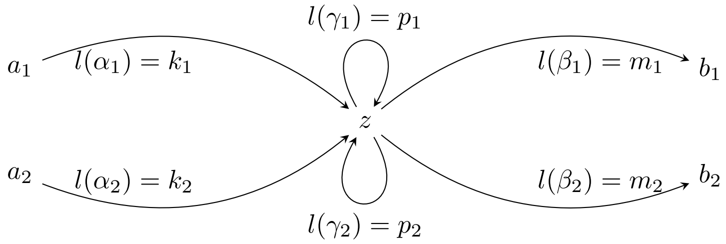

FIGURE 1.

4.3. **Equivalence of (C).** For this part of the proof we will need to recall the notion of weak incompressibility introduced by Šarkovs′kiĭ i in [20].

A subset Λ of X is *weakly incompressible* if $M \cap Cl_X(f(\Lambda \setminus M)) \neq \emptyset$ whenever M is a non-empty, closed, proper subset of Λ. Clearly Λ is weakly incompressible if and only if $Cl_X(f(U)) \setminus (\Lambda \setminus U) \neq \emptyset$ for any non-empty proper subset U of Λ which is open in Λ. For example, every ω-limit set (see [2]) is weakly incompressible, in particular, if Λ is a finite orbit, then Λ is weakly incompressible. In [2, Theorem 2.2] Barwell et al. prove that the notion of weak incompressibility coincides with that of internal chain transitivity. In case $\Lambda = X$, we also say that the map f is weakly incompressible, or equivalently, chain transitive. A close inspection of the proof of Theorem 2.2 in [2] reveals that, due to compactness, it suffices to verify weak incompressibility condition only on the elements of some basis for topology on X.

Lemma 9. *Let X be a compact metric space and let $f\colon X \to X$ be a continuous function. Let B^* a base for X and let $B = B^* \setminus \{\emptyset, X\}$. Then X is weakly incompressible if and only if for every $U \in B$, $Cl_X(f(U)) \cap X \setminus U \neq \emptyset$.*

We now proceed to prove (C) $\implies$ (A3). First note that if for any non-empty proper basic set $\langle U, V\rangle_2$ we have that $Cl_{F_2(X)}(f_2(\langle U, V\rangle_2)) \cap F_2(X) \setminus \langle U, V\rangle_2 \neq \emptyset$, then, by Lemma 9, $F_2(X)$ is weakly incompressible, and thus f_2 is chain transitive and (A3) holds. For this reason, in one direction, it suffices to prove that (C) implies

$$Cl_{F_2(X)}(f_2(\langle U, V\rangle_2)) \cap F_2(X) \setminus \langle U, V\rangle_2 \neq \emptyset, \text{ for any proper basic set } \langle U, V\rangle_2 \neq \emptyset.$$

To that end, let $\langle U, V\rangle_2$ be a non-empty proper basic set of $F_2(X)$. Assume first that $U \cap V \neq \emptyset$. Since f is chain transitive, $Cl_X(f(U \cap V)) \cap X \setminus U \cap V \neq \emptyset$. Let $x \in Cl_X(f(U \cap V)) \cap X \setminus U \cap V$. It is clear that $\{x\} \in Cl_{F_2(X)}(f_2(\langle U, V\rangle_2)) \cap F_2(X) \setminus \langle U, V\rangle_2$.

Assume now that $U \cap V = \emptyset$. Notice that U and V are non-empty proper open subsets of X. Thus, we have $Cl_X(f(U)) \cap X \setminus U \neq \emptyset$ and $Cl_X(f(V)) \cap X \setminus V \neq \emptyset$. Let $x \in Cl_X(f(U)) \cap X \setminus U$ and $y \in Cl_X(f(V)) \cap X \setminus V$. It is easy to see that $\{x, y\} \in Cl_{F_2(X)}(f_2\langle U, V\rangle_2)$. If $x = y$, then $\{x\} \notin \langle U, V\rangle_2$, then $\{x\} \in Cl_{F_2(X)}(f_2(\langle U, V\rangle_2)) \cap F_2(X) \setminus \langle U, V\rangle_2$. If $x \neq y$ and $\{x, y\} \in \langle U, V\rangle_2$, then $x \in V$ and $y \in U$. Without loss of generality, assume that $Cl_X(f(U)) \nsubseteq V$. Let $z \in Cl_X(f(U)) \setminus V$. Since $y \in Cl_X(f(V)) \setminus V$, we have that $\{z, y\} \in Cl_{F_2(X)}(f_2(\langle U, V\rangle_2))$ and $\{z, y\} \notin \langle U, V\rangle_2$. This proves that $Cl_{F_2(X)}(f_2(\langle U, V\rangle_2)) \cap F_2(X) \setminus \langle U, V\rangle_2 \neq \emptyset$.

For the converse we prove (B2) $\implies$ (C). For the sake of contradiction, assume that (C) does not hold. As we are assuming (B2), f must be CT, and hence there exist two non-empty proper disjoint open subsets of X, U and V, such that $Cl_X(f(U)) \subseteq V$ and $Cl_X(f(V)) \subseteq U$. By regularity we can find a $\delta > 0$ such that

$N_X(f(U),\delta) \subseteq V$ and $N_X(f(V),\delta) \subseteq U$. Let $z \in X$ be any point in U. By (B2) we can find two δ-chains from z to itself of co-prime lengths. Note however that any δ-chain starting in U must after one step end up in V, then after another again in U etc. This shows that any δ-chain from z to itself must be of even length, a contradiction.

4.4. **Equivalence of (D1)-(D2).** This easily follows from Proposition 11 which will allow us to conclude (A3) $\iff$ (D1) and (A3) $\iff$ (D2). But first we need a lemma.

Lemma 10. *Let X be a compact metric space, let Y be a metric space and let $f\colon X \to X$, and $g\colon Y \to Y$ be continuous functions. Suppose that there exists $h\colon X \to Y$ onto and continuous such that $h \circ f = g \circ h$. If f is chain transitive, then g is chain transitive.*

Proof. We will denote d_1 and d_2 to the metrics on X and Y, respectively. Let $y_1, y_2 \in Y$ and $\varepsilon > 0$. Since h is onto, there are x_1 and x_2 in X such that $h(x_1) = y_1$ and $h(x_2) = y_2$. Also, since X is compact, there is $\delta > 0$ such that if $d_1(a,b) < \delta$, then $d_2(h(a),h(b)) < \varepsilon$, for all $a, b \in X$. By hypothesis, there is a δ-chain in X, $\langle x_1 = z_0, z_1, \ldots, z_r = x_2\rangle$, with $r \geq 1$. Thus, for $i \in \{0,1,\ldots,r-1\}$, we have that $d_1(f(z_i), z_{i+1}) < \delta$, then $d_2(g(h(z_i)), h(z_{i+1})) = d_2(h(f(z_i)), h(z_{i+1})) < \varepsilon$. Hence, the sequence $\langle y_1 = h(z_0), h(z_1), \ldots, h(z_r) = y_2\rangle$ is an ε-chain in Y. □

Proposition 11. *Let X be a compact metric space, let $f\colon X \to X$ be a continuous function and let $n \geq 1$. The following are equivalent:*

1. *$f^{(n)}\colon X^{(n)} \to X^{(n)}$ is chain transitive,*
2. *$f_n\colon F_n(X) \to F_n(X)$ is chain transitive.*

Proof. Let $h\colon X^{(n)} \to F_n(X)$ given by: for each $(x_1, x_2, \ldots, x_n)$ in $X^{(n)}$,

$$h((x_1, x_2, \ldots, x_n)) = \{x \in X\colon x = x_i, \text{ for some } i \in \{1,2,\ldots,n\}\}.$$

It is not difficult to see that h is continuous. Also, if $(x_1, x_2, \ldots, x_n) \in X^{(n)}$, then

$$\begin{aligned}(h \circ f^{(n)})((x_1, x_2, \ldots, x_n)) &= h((f(x_1), f(x_2), \ldots, f(x_n))) = \\ &\{x \in X\colon x = f(x_i), \text{ for some } i \in \{1,2,\ldots,n\}\} = \\ &f_n(\{x \in X\colon x = x_i, \text{ for some } i \in \{1,2,\ldots,n\}\}) = \\ &f_n(h((x_1, x_2, \ldots, x_n))) = (f_n \circ h)((x_1, x_2, \ldots, x_n)).\end{aligned}$$

Thus, by Lemma 10, we have that (1) implies (2).

To see that (2) implies (1), let $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^{(n)}$ and let $\varepsilon > 0$. Let $z \in X$, then $h((x_1, x_2, \ldots, x_n)), h((y_1, y_2, \ldots, y_n))$ and $\{z\}$ are in $F_n(X)$. By hypothesis there are ε-chains in $F_n(X)$ in the following way:

$$\langle h((x_1, x_2, \ldots, x_n)) = A_0, A_1, \ldots, A_{m_1} = \{z\}\rangle \text{ and}$$
$$\langle \{z\} = B_0, B_1, \ldots, B_{m_2} = h((y_1, y_2, \ldots, y_n))\rangle.$$

For each $i \in \{1,2,\ldots,n\}$, we have induced ε-chains in X, $\left\langle x_i = a_0^i, a_1^i, \ldots, a_{m_1}^i = z\right\rangle$ and $\left\langle z = b_0^i, b_1^i, \ldots, b_{m_2}^i = y_i\right\rangle$, where $a_j^i \in A_j$ and $b_t^i \in B_t$, for each $j \in \{0,1,\ldots,m_1\}$ and each $t \in \{0,1,\ldots,m_2\}$. Thus, the sequence

$$\langle D_0, D_1, \ldots, D_{m_1}, D_{m_1+1}, \ldots, D_{m_1+m_2}\rangle,$$

given by:

$$D_i = \begin{cases} (a_i^1, a_i^2, \ldots, a_i^n) & \text{if } i \in \{0,1,\ldots,m_1\}; \\ (b_{i-m_1}^1, b_{i-m_1}^2, \ldots, b_{i-m_1}^n) & \text{if } i \in \{m_1+1, m_1+2, \ldots, m_1+m_2\} \end{cases}$$

is an ε-chain in $X^{(n)}$ from $(x_1, x_2, \ldots, x_n)$ to $(y_1, y_2, \ldots, y_n)$. □

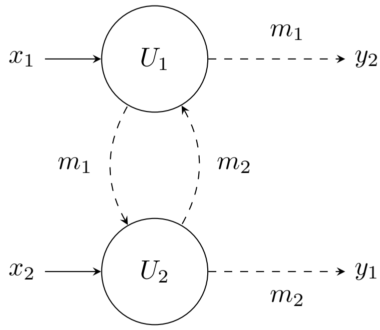

FIGURE 2.

4.5. **Equivalence of (E1)-(E3).** First note that (E1) $\implies$ (D1) $\iff$ (D2).

We next show that (D2) $\implies$ (E3). Let $U \subseteq X$ be a non-empty open set and let $\varepsilon > 0$. Since X is compact, there are $x_1, x_2, \ldots, x_k \in X$ such that $X = \bigcup_{i=1}^{k} N_X(x_i, \frac{\varepsilon}{2})$. By (D2), $f^{(k)}\colon X^{(k)} \to X^{(k)}$ is chain transitive. Let $u \in U$. Since $(u, u, \ldots, u), (x_1, x_2, \ldots, x_k) \in X^{(k)}$, there is an $\frac{\varepsilon}{2}$-chain

$$\Gamma = \langle (u, u, \ldots, u) = (z_1^0, z_2^0, \ldots, z_k^0), (z_1^1, z_2^1, \ldots, z_k^1), \ldots \\ \ldots, (z_1^r, z_2^r, \ldots, z_k^r) = (x_1, x_2, \ldots, x_k) \rangle$$

Let $x \in X$, then $x \in N_X(x_j, \frac{\varepsilon}{2})$ for some $j \in \{1, 2, \ldots, k\}$. Thus, $d\left(f(z_j^{r-1}), x\right) < d\left(f(z_j^{r-1}), x_j\right) + d\left(x_j, x\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore, the sequence:

$$\left\langle u = z_j^0, z_j^1, \ldots, z_j^{r-1}, x \right\rangle$$

is an ε-chain from u to x of length k. Hence, (E3) holds.

We now prove (E3) $\implies$ (E2). Let $\varepsilon > 0$ and let U be a non-empty open subset of X. Since U is non-empty, let $u \in U$. By hypothesis, there is a positive integer $n_\varepsilon \geq 1$ such that for each $x \in X \setminus \{u\}$, there is an ε-chain of length n_ε from u to x as desired.

Finally to close the circle of equivalences we prove (E2) $\implies$ (E1). Suppose that f is exact by chains. To see that $f^{(2)}\colon X^{(2)} \to X^{(2)}$ is chain transitive, let $(x_1, x_2), (y_1, y_2) \in X^{(2)}$ and let $\varepsilon > 0$. Let $U_1 = N_X\left(f(x_1), \varepsilon\right)$ and $U_2 = N_X\left(f(x_2), \varepsilon\right)$. By hypothesis, there are positive integers m_1, m_2 such that for each $x \in X$, there exist $u_1 \in U_1$, $u_2 \in U_2$, and ε-chains, $\langle u_1 = a_0, a_1, \ldots, a_{m_1} = x\rangle$ and $\langle u_2 = b_0, b_1, \ldots, b_{m_2} = x\rangle$. Thus, for $f(x_2)$, there are $z_1 \in U_1$ and an ε-chain $\langle z_1 = c_0, c_1, \ldots, c_{m_1} = f(x_2)\rangle$. Analogously, for $f(x_1)$, there are $z_2 \in U_2$ and an ε-chain $\langle z_2 = d_0, d_1, \ldots, d_{m_2} = f(x_1)\rangle$. Then, for y_1, we have $z_2' \in U_2$ and an ε-chain $\langle z_2' = s_0, s_1, \ldots, s_{m_2} = y_1\rangle$. Similarly, for y_2, there are $z_1' \in U_1$ and an ε-chain $\langle z_1' = t_0, t_1, \ldots, t_{m_1} = y_2\rangle$. Thus, the sequences:

$$\langle x_1, z_1 = c_0, c_1, \ldots, c_{m_1} = f(x_2), z_2' = s_0, s_1, \ldots, s_{m_2} = y_1\rangle \text{ and}$$

$$\langle x_2, z_2 = d_0, d_1, \ldots, d_{m_2} = f(x_1), z_1' = t_0, t_1, \ldots, t_{m_1} = y_2\rangle$$

are ε-chains of the same length. Thus, the sequence:

$$\langle (x_1, x_2), (z_1, z_2) = (c_0, d_0), \ldots, (s_{m_2}, t_{m_1}) = (y_1, y_2)\rangle$$

is an ε-chain in $X^{(2)}$ (see Figure 2).

4.6. **Equivalence of (F1)-(F2).** In [19, Corollary 12] Richeson et al. prove that (F1) $\iff$ (F2). It therefore remains to prove the equivalence of, say (F1) to one of the statements above.

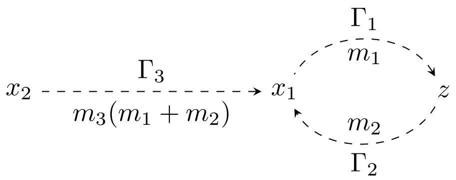

FIGURE 3.

We first show (D2) $\implies$ (F1). Let $n \geq 1$ and $\varepsilon > 0$. Since X is compact, there is a sequence $0 < \delta_1 < \delta_2 < \ldots < \delta_n = \varepsilon$ such that if $d(a,b) < \delta_i$, then $d\left(f(a), f(b)\right) < \frac{\delta_{i+1}}{2}$, for each $i \in \{1, 2, \ldots, n-1\}$. Let $\delta = \frac{\delta_1}{2}$. By (D2), there is a δ-chain in $X^{(n)}$

$$\langle (x, y, \ldots, y) = (z_1^0, z_2^0, \ldots, z_n^0), (z_1^1, z_2^1, \ldots, z_n^1), \ldots \\ \ldots, (z_1^t, z_2^t, \ldots, z_n^t) = (y, y, \ldots, y)\rangle$$

with $t \geq 1$. Thus, the sequence:

$$\langle x = z_1^0, z_1^1, z_1^2, \ldots, z_1^t = y = z_2^0, z_2^1, \ldots, z_2^t = y = z_3^0, \ldots, z_n^t = y\rangle$$

is a δ-chain of length tn. Renaming the elements of this δ-chain, we can write it as:

$$\langle x = a_0, a_1, \ldots, a_{tn} = y\rangle$$

Since $d\left(f(x), a_1\right) < \delta < \delta_1$, then $d\left(f^2(x), f(a_1)\right) < \frac{\delta_2}{2}$. Also we have that $d\left(f(a_1), a_2\right) < \frac{\delta_2}{2}$, then $d\left(f^2(x), a_2\right) < \delta_2$. Again, since $d\left(f^2(x), a_2\right) < \delta_2$, then $d\left(f^3(x), f(a_2)\right) < \frac{\delta_3}{2}$. But $d\left(f(a_2), a_3\right) < \frac{\delta_3}{2}$, then $d\left(f^3(x), a_3\right) < \delta_3$. Continue this process, we finally get $d\left(f^n(x), a_n\right) < \delta_n = \varepsilon$. Analogously, $d\left(f^n(a_{in}), a_{(i+1)n}\right) < \varepsilon$, for each $i \in \{0, 1, \ldots, t-1\}$. Thus, the sequence:

$$\langle x = z_0, z_1 = a_n, z_2 = a_{2n}, \ldots, z_t = a_{tn} = y\rangle$$

is an ε-chain in X for the function f^n. Hence, f^n is chain transitive for every $n \geq 1$.

Finally, we show (F1) $\implies$ (E1). Let $(x_1, x_2), (y_1, y_2) \in X^{(2)}$, let $\varepsilon > 0$ and let $z \in X$. Since f is totally chain transitive, in particular f is chain transitive. Then there is an ε-chain Γ_1, of length m_1, from x_1 to z and also there is an ε-chain Γ_2, of length m_2, form z to x_1. By hypothesis, the function $f^{m_1+m_2}$ is chain transitive, then there is an ε-chain Γ_3' for $f^{m_1+m_2}$ from x_2 to x_1 of length m_3, $\Gamma_3' = \langle x_2 = a_0, a_1, \ldots, a_{m_3} = x_1\rangle$. Since the sequence Γ_3 given by:

$$\Gamma_3 = \langle x_2 = a_0, f(a_0), f^2(a_0), \ldots, f^{m_1+m_2-1}(a_0), a_1, f(a_1), \ldots \\ \ldots, f^{m_1+m_2-1}(a_1), a_2, \ldots, a_{m_3-1}, f(a_{m_3-1}), \ldots \\ \ldots, f^{m_1+m_2-1}(a_{m_3-1}), a_{m_3} = x_1\rangle$$

is an ε-chain for f from x_2 to x_1 of length $m_3(m_1 + m_2)$. Hence, the sequences $\Gamma_1 + m_3(\Gamma_2 + \Gamma_1)$ and $\Gamma_3 + \Gamma_1$ are ε-chains from x_1 to z and from x_2 to z, respectively, of length $m_3(m_1 + m_2) + m_1$ (see Figure 3). Thus, these ε-chains of the same length induce an ε-chain Γ in $X^{(2)}$ from (x_1, x_2) to (z, z). Analogously, we can construct an ε-chain Ψ in $X^{(2)}$ from (z, z) to (y_1, y_2). Therefore, the concatenation $\Gamma + \Psi$ is an ε-chain in $X^{(2)}$ from (x_1, x_2) to (y_1, y_2). Hence, $f^{(2)} \colon X^{(2)} \to X^{(2)}$ is chain transitive. This finishes the proof of our main theorem.

5. AN APPLICATION TO THE DYNAMICS ON CONTINUA

Recall that a *continuum* is a non-empty compact connected metric space.

Lemma 12. *Let X be a continuum and let $f\colon X \to X$ be a continuous function. Suppose that f is chain transitive. If U and V are two non-empty proper disjoint open subsets of X, then either $Cl_X(f(U)) \nsubseteq V$ or $Cl_X(f(V)) \nsubseteq U$.*

Proof. Assume that the lemma is not true. Let U, V be non-empty proper disjoint open subsets of X such that $Cl_X(f(U)) \subset V$ and $Cl_X(f(V)) \subset U$. Let $\varepsilon_1, \varepsilon_2 > 0$ such that $N_X(Cl_X(f(U)), \varepsilon_1) \subset V$ and $N_X(Cl_X(f(V)), \varepsilon_2) \subset U$. Since X is connected, there is $z \in X \setminus (U \cup V)$. If $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, then for $x \in U$, $f(x) \in Cl(f(U))$. Thus, if $x_1 \in X$ is such that $d(x_1, f(x)) < \varepsilon$, then $x_1 \in V$ and, thus, $f(x_1) \in Cl(f(V))$. If $x_2 \in X$ is such that $d(x_2, f(x_1)) < \varepsilon$, then $x_2 \in U$. Therefore, there is no an ε-chain from x to z, which contradicts our hypothesis. □

As a consequence of Lemma 12 and Theorem 3 we have the following corollary.

Corollary 13. *Let X be a continuum and $f\colon X \to X$ a continuous function. Then f is chain transitive if and only if one, and hence all of the conditions in Theorem 3 hold. Furthermore all of these conditions are equivalent to requiring that f is chain recurrent.*

Proof. For the last part recall that in [19, Corollary 14] Richeson et al. prove that, for continua, chain recurrence is equivalent to chain transitivity. □

Using the fact that if X is a continuum, then 2^X is also a continuum (see [15, Corollary 1.8.9]), and the equivalence between chain transitivity of f and 2^f from the previous corollary, we obtain the following result.

Corollary 14. *Let X be a continuum and let $f\colon X \to X$ be a continuous function. The following are equivalent:*

(1) *f is chain transitive,*
(2) *2^f is chain transitive,*
(3) *2^f is totally chain transitive,*
(4) *2^f is chain weakly mixing,*
(5) *2^f exact by chains,*
(6) *2^f is chain mixing,*
(7) *2^f is chain recurrent.*

Corollary 13 tells us that in continua, if f is chain transitive, then f_n and 2^f are chain transitive. This does not hold for $C(f)$ in general. In the following example we give a chain transitive continuous function which has a fixed point and thus, by Corollary 5, 2^f is chain transitive but $C(f)$ is not.

Example 4. Let $I = [0,1]$ be the unit interval and let $T\colon I \to I$ be the tent map as in Example 2. We have seen that T is weakly mixing, hence transitive, and in particular chain transitive. Let $A = [0,1]$ and $B = \{0\}$. Assume that there is a $\frac{1}{8}$-chain in $C(X)$ from A to B, $\Gamma = \langle A = C_0, C_1, C_2, \ldots, C_k = B\rangle$, with $k \in \mathbb{N}$. Since $H(C(T)(A), C_1) < \frac{1}{8}$, then $[0,1] \subseteq N(C_1, \frac{1}{8})$. Then $\left[\frac{1}{8}, \frac{7}{8}\right] \subseteq C_1$, thus, $\left[\frac{1}{4}, 1\right] = C(T)\left(\left[\frac{1}{8}, \frac{7}{8}\right]\right) \subseteq C(T)(C_1)$. Now, since $H(C(T)(C_1), C_2) < \frac{1}{8}$, then $\left[\frac{1}{4}, 1\right] \subseteq C(T)(C_1) \subset N\left(C_2, \frac{1}{8}\right)$, and thus, $\left[\frac{3}{8}, \frac{7}{8}\right] \subseteq C_2$. Therefore, $\left[\frac{1}{4}, 1\right] = C(T)\left(\left[\frac{3}{8}, \frac{7}{8}\right]\right) \subseteq C(T)(C_2)$. We can see that $\left[\frac{1}{4}, 1\right]$ is always a subset of $C(T)(C_j)$, for each $j \in \{0, 1, \ldots, k-1\}$. Thus, $H(C(T)(C_{k-1}), B)$ can not be less that $\frac{1}{8}$. This implies that $C(T)$ is not chain transitive.

Theorem 15. *Let X be a compact metric space and let $f\colon X \to X$ be a continuous function such that f_2 is chain transitive. If X has finitely many components, then X is connected.*

Proof. Assume that X has r components say $K_1, K_2, \dots, K_r$. Let $\delta > 0$ such that $\delta < \min \{d(K_i, K_j) \mid i, j \in 1, 2, \dots, r\}$. If $A = \{a_1, a_2\} \subseteq K_1$ and $B = \{b_1, b_2\}$ such that $b_i \in K_i$, $i \in \{1, 2\}$, then there is no a δ-chain from A to B in $F_2(X)$, which contradicts our assumption. $\square$

Nevertheless, it is possible to find a space X, which has infinitely many components, and a continuous function $f\colon X \to X$ for which $f_2\colon F_2(X) \to F_2(X)$ is chain transitive.

Example 5. Let $\Sigma_2 = \prod_{i=1}^{\infty} A_i$, where $A_i = \{0, 1\}$ for all $i \geq 1$, and let $\sigma\colon \Sigma_2 \to \Sigma_2$ given by $\sigma(a_1, a_2, ...) = (a_2, a_3, ...)$. The function σ is known as the *shift map*. It is easy to see that $\sigma\colon \Sigma_2 \to \Sigma_2$ is exact and, therefore, weakly mixing. Thus, $\sigma_n\colon F_n(\Sigma_2) \to F_n(\Sigma_2)$ is transitive for each $n \geq 1$ [10, Theorem 4.5], thus σ and σ_n are chain transitive.

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