Combinatorial Settlement Model: Resistance to Predators and Altruists

5th Croatian Combinatorial Days
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Our preprint

PREDATORS AND ALTRUISTS ARRIVING ON JAMMED RIVIERA

TOMISLAV DOŠLIĆ, MATE PULJIZ, STJEPAN ŠEBEK, AND JOSIP ŽUBRINIĆ

Abstract. The Riviera model is a combinatorial model for a settlement along a coastline, introduced recently by the authors. Of most interest are the so-called jammed states, where no more houses can be built without violating the condition that every house needs to have free space to at least one of its sides. In this paper, we introduce new agents (predators and altruists) that want to build houses once the settlement is already in the jammed state. Their behavior is governed by a different set of rules, and this allows them to build new houses even though the settlement is jammed. Our main focus is to detect jammed configurations that are resistant to predators, to altruists, and to both predators and altruists. We provide bivariate generating functions, and complexity functions (configurational entropies) for such jammed configurations. We also discuss this problem in the two-dimensional setting of a combinatorial settlement planning model that was also recently introduced by the authors, and of which the Riviera model is just a special case.

1. Introduction

https://arxiv.org/pdf/2401.01225

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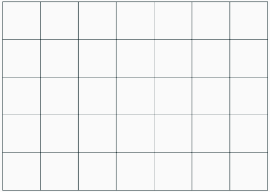

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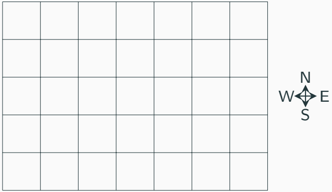

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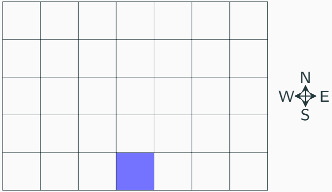

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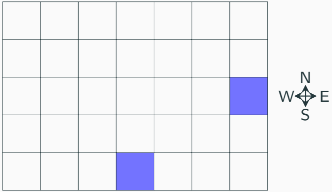

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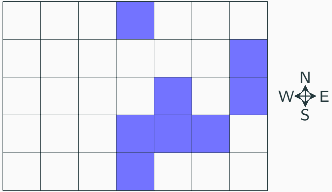

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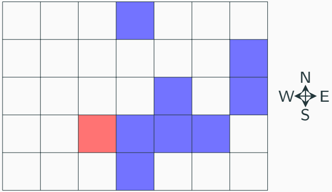

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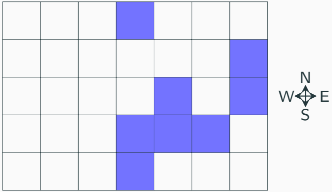

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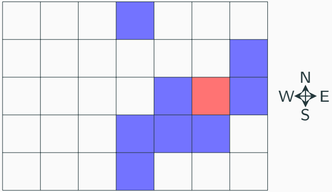

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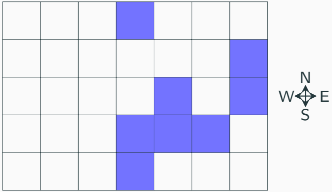

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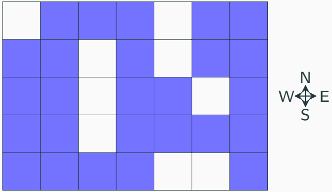
- Sunlight comes from **East**, **South** and **West**
- Build houses ensuring that **each is adjacent to at least one unoccupied lot to its East, South, or West**
- Continue adding houses to empty lots until reaching a **jammed configuration**



Key questions

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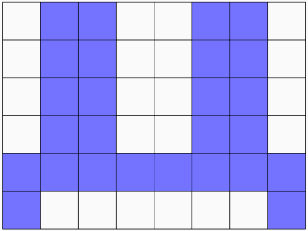
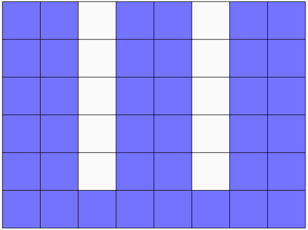
- Distribution over possible densities ρ? (under which model?)
- What is the **jamming limit**, i.e., the average density of jammed configurations?
- How does the behavior change when sampling under different models?

What can be the occupancy of jammed configurations?



An inefficient configuration ($\rho \approx \frac{1}{2}$)



An efficient configuration ($\rho \approx \frac{3}{4}$)

Summary of known results ($\frac{1}{2} \leq \rho \leq \frac{3}{4}$)

Proposition (PŠŽ)
Inefficient jammed configurations on $m \times n$ grid have the following occupancy:

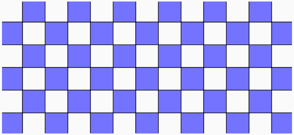
$$I_{m,n} = \begin{cases} \frac{mn}{2} + 2, & \text{if } n \equiv 0 \pmod 4, \\ \frac{m(n+2)}{2}, & \text{if } n \equiv 2 \pmod 4, \\ \frac{m(n+1)}{2} + 1, & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Proposition (PŠŽ)
The occupancy of efficient jammed configurations on $m \times n$ grid, $m, n \geq 2$, satisfies:

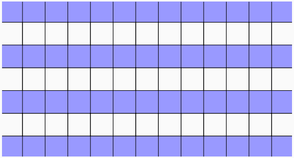
$$E_{m,n} \leq \begin{cases} mn - \left\lfloor \frac{n}{4} \right\rfloor \cdot (m-1), & \text{if } n \not\equiv 3 \pmod 4, \\ mn - \left\lfloor \frac{n}{4} \right\rfloor \cdot (m-1) - \left\lfloor \frac{m}{2} \right\rfloor, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

▷ P., Šebek and Žubrinić, Combinatorial settlement planning, *Contrib. Discrete Math.* **18** (2023), no. 2, 20–47

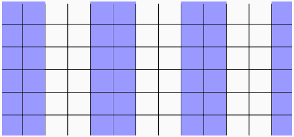
Extremal infinite grid patterns



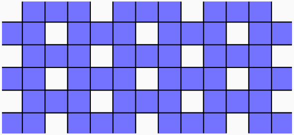
Check pattern ($\rho \approx \frac{1}{2}$)



Stripe pattern ($\rho \approx \frac{1}{2}$)



Rake pattern ($\rho \approx \frac{1}{2}$)



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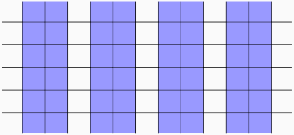
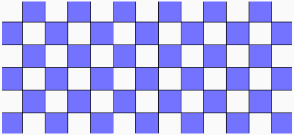


Comb pattern ($\rho \approx \frac{3}{4}$)

Extremal infinite grid patterns



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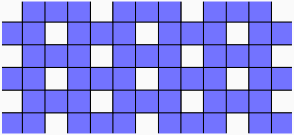

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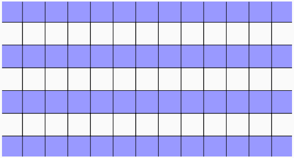

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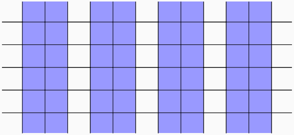

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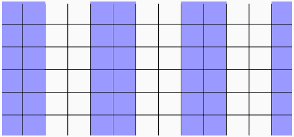


Rake pattern ($\rho \approx \frac{1}{2}$)

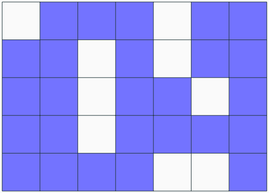
- combine them to get densities in-between
- take a bounded piece to build a finite grid

Flashback to CroCoDays 2022

Predators and altruists

Introducing predators and altruists in the mix

- Predators and altruists are happy to build further in jammed configurations.



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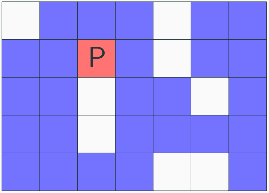
- **Predators**: build new houses if they receive sunlight, even if it blocks others.



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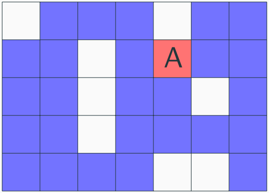

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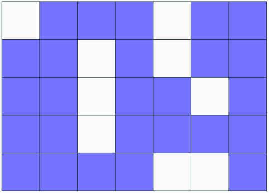
- **Predators**: build new houses if they receive sunlight, even if it blocks others.
- **Altruists**: avoid blocking sunlight to others, but may build even if they don't receive sunlight.
- Question: How do jammed configurations which are resistant to these new asocial agents look like?



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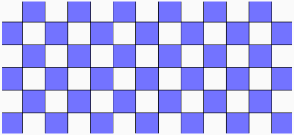
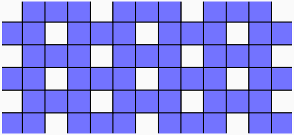
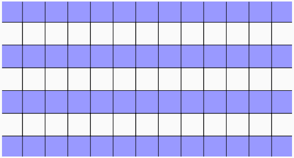


Check pattern ($\rho \approx \frac{1}{2}$) AR✗ PR✓



Brick pattern ($\rho \approx \frac{3}{4}$) AR✓ PR✓



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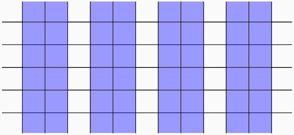
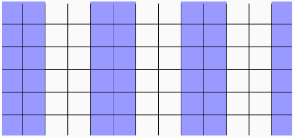


Comb pattern ($\rho \approx \frac{3}{4}$) AR✓ PR✗



Rake pattern ($\rho \approx \frac{1}{2}$) AR✓ PR✗

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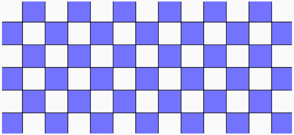

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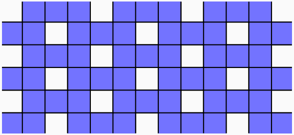

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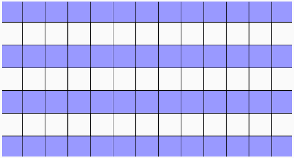

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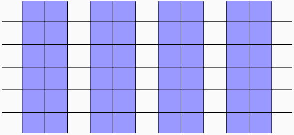

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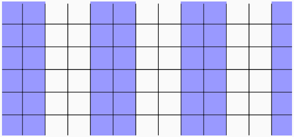

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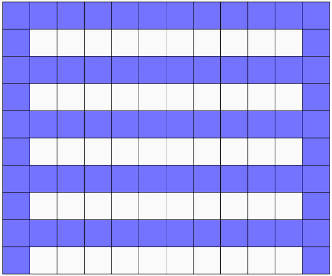
- all but Check pattern are resistant to Altruists
- just Check and Brick patterns are resistant to Predators

Jammed configurations resistant to Altruists

E.g.



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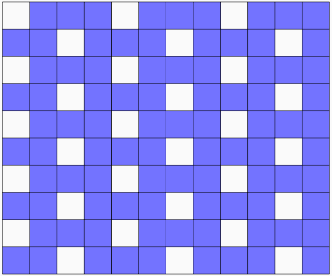

Brick pattern $\rho \approx \frac{3}{4}$

$$\frac{1}{2} \leq \rho \leq \frac{3}{4}$$

Jammed configurations resistant to Predators

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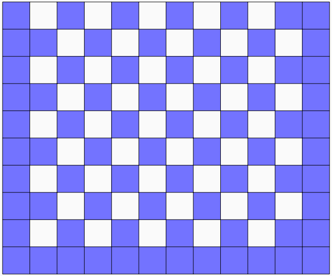
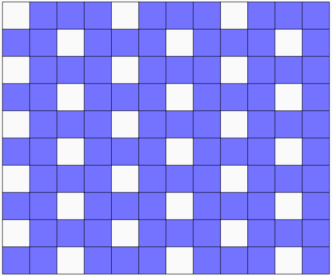


Check pattern $\rho \approx \frac{1}{2}$



Brick pattern $\rho \approx \frac{3}{4}$

$$\frac{1}{2} \leq \rho \leq ??$$

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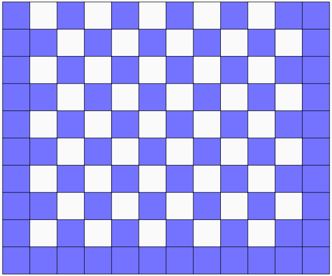

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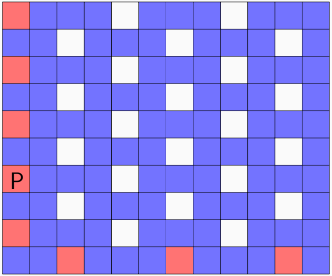

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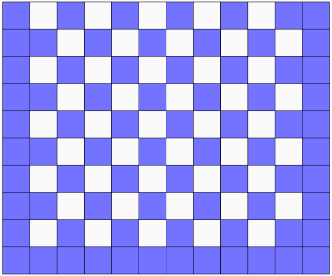

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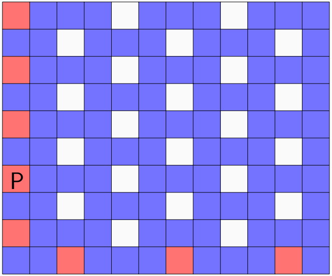
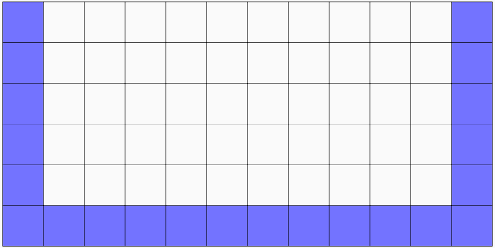


Brick pattern $\rho \approx \frac{3}{4}$ ✗

$$\frac{1}{2} \leq \rho \leq \mathbf{\frac{2}{3}}$$

Proof that $\rho \leq \frac{2}{3}$ for PR jammed configurations

Fact 0: border must be bricked up



Proof that $\rho \leq \frac{2}{3}$ for PR jammed configurations

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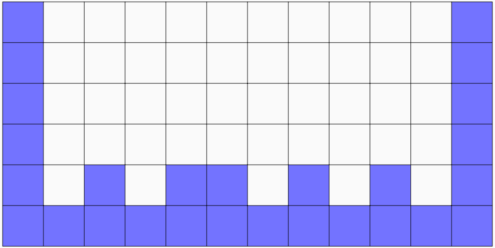
Fact 1: occupancy of the penultimate row $\leq 2\left\lceil \frac{\#\text{ columns}}{3} \right\rceil$



no two successive empty lots, at most two successive occupied $\implies$ at least one empty on every three lots

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Fact 0: border must be bricked up

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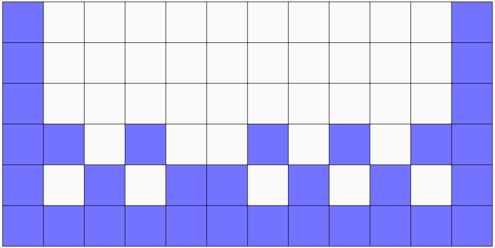
Fact 2: occupancy in successive rows (going up) can increase by at most 1



no two adjacent empty lots

Proof that $\rho \leq \frac{2}{3}$ for PR jammed configurations

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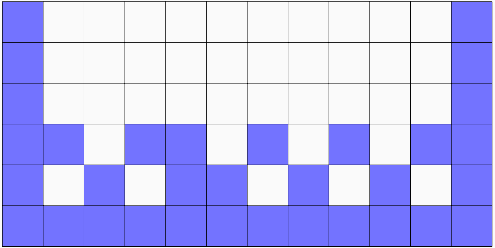
Fact 1: occupancy of the penultimate row $\leq 2 \left\lceil \frac{\text{\# columns}}{3} \right\rceil$

Fact 2: occupancy in successive rows (going up) can increase by at most 1



no gaps can be completely filled up $\implies$ number of empty lots can decrease by at most one

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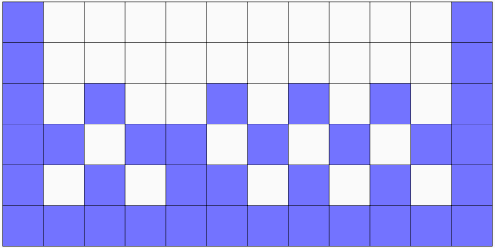
Fact 3: if it increased by 1, then it must decrease by at least 1 in the next row up



no two adjacent empty lots

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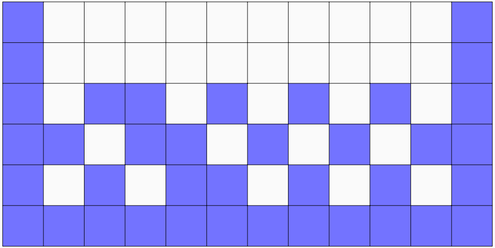

in this case lots in columns 2 and $n-1$ must be empty, and one more in each gap $\implies$ number of empty lots increased by at least one

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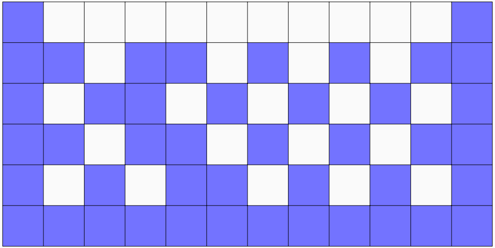

etc.

□

Is this bound attained?

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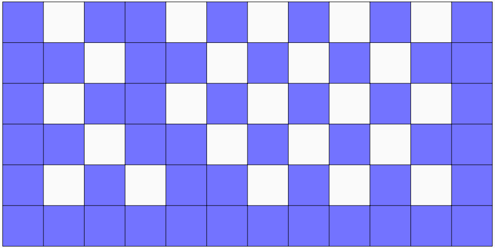


etc.

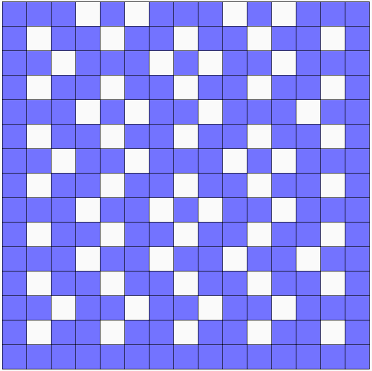
□

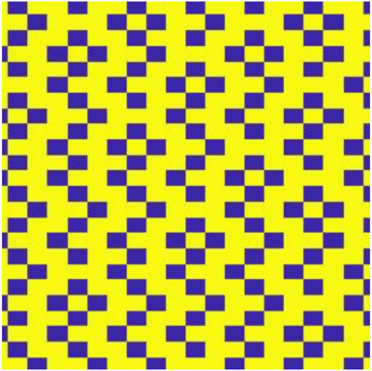
Is this bound attained?

Evolutionary stable configurations (ESC)

Evolutionary stable configurations – rare flowers

Evolutionary Stable Configurations: resistant to both predators and altruists.





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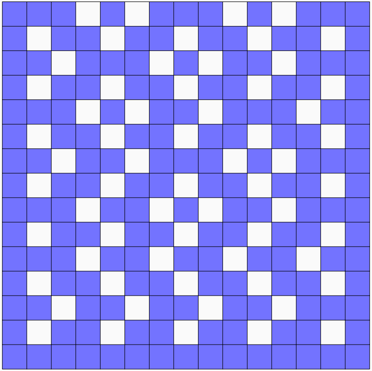

Not obvious, surprising rigid structure, $\rho \approx \frac{2}{3}$, exist only for certain grid dimensions,...

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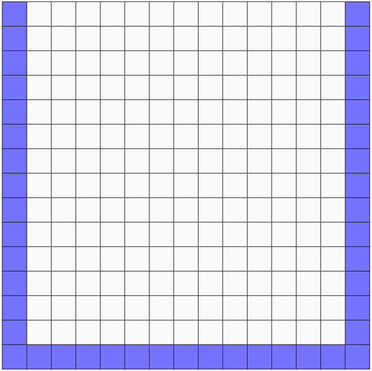

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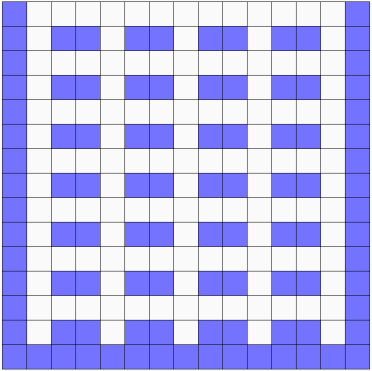

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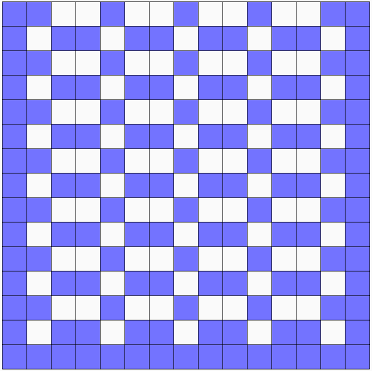

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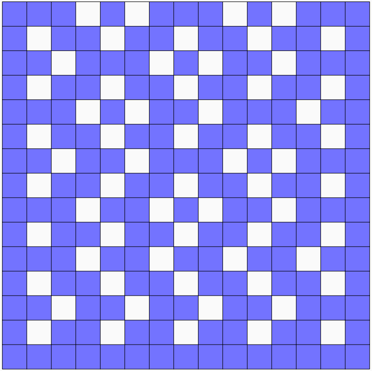
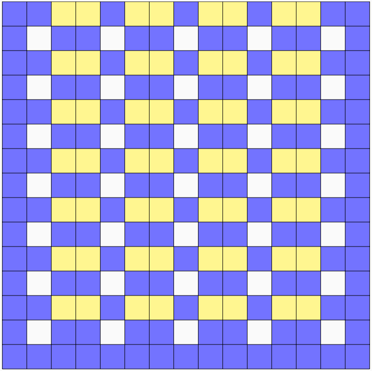


Not obvious, surprising rigid structure, $\rho \approx \frac{2}{3}$, exist only for certain grid dimensions,...

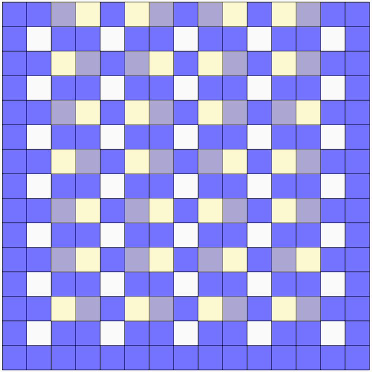
Theorem

If $m \times n$ ESC exists for $m, n > 2$, then n is divisible by 3 and m is odd, and it must have the structure below:



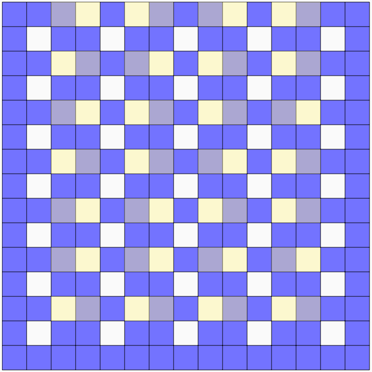
where exactly one in each highlighted pair of adjacent lots is occupied, and the other is empty. As a consequence, all the ES configurations have the same occupancy of $mn - \frac{(m-1)(2n-3)}{6} = \frac{2}{3}mn + \frac{1}{2}(m-1) + \frac{1}{3}n$.

Choice is not completely arbitrary



$$\overset{encode}{\leftrightarrow} \begin{bmatrix} R & L & R & L \\ L & R & L & L \\ R & L & L & R \\ L & L & R & L \\ R & R & L & L \\ R & R & R & R \\ L & L & L & L \end{bmatrix}$$

Choice is not completely arbitrary



$$\overset{encode}{\leftrightarrow} \quad \begin{bmatrix} R & L & R & L \\ L & R & L & L \\ R & L & L & R \\ L & L & R & L \\ R & R & L & L \\ R & R & R & R \\ L & L & L & L \end{bmatrix}$$

Theorem

Allowed ESCs are precisely those which (when encoded) do not contain any of the forbidden constellations:

$$\begin{matrix} R & * \\ R & L \end{matrix} \quad , \quad \begin{matrix} * & L \\ R & L \end{matrix} \quad , \quad \left| \begin{matrix} L \\ L \end{matrix} \right. \quad , \quad \left. \begin{matrix} R \\ R \end{matrix} \right| \quad .$$

Our preprint

PREDATORS AND ALTRUISTS ARRIVING ON JAMMED RIVIERA

TOMISLAV DOŠLIĆ, MATE PULJIZ, STJEPAN ŠEBEK, AND JOSIP ŽUBRINIĆ

Abstract. The Riviera model is a combinatorial model for a settlement along a coastline, introduced recently by the authors. Of most interest are the so-called jammed states, where no more houses can be built without violating the condition that every house needs to have free space to at least one of its sides. In this paper, we introduce new agents (predators and altruists) that want to build houses once the settlement is already in the jammed state. Their behavior is governed by a different set of rules, and this allows them to build new houses even though the settlement is jammed. Our main focus is to detect jammed configurations that are resistant to predators, to altruists, and to both predators and altruists. We provide bivariate generating functions, and complexity functions (configurational entropies) for such jammed configurations. We also discuss this problem in the two-dimensional setting of a combinatorial settlement planning model that was also recently introduced by the authors, and of which the Riviera model is just a special case.

1. Introduction

https://arxiv.org/pdf/2401.01225

Thank You!

Forbidden constellations

$$\begin{array}{cccc|ccc|c|ccc|ccc} 0 & * & * & * & * & * & * & \dots & * & * & * & * & * & * \\ * & \underline{0} & * & * & * & * & * & \dots & * & * & * & 1 & 1 & 1 \\ * & * & 0 & 1 & 1 & 0 & 1 & \dots & 1 & 0 & 1 & 1 & 0 & 1 \end{array}$$

(a) Type I

$$\begin{array}{cccc|ccc|c|ccc|ccc} * & \underline{0} & * & * & * & * & * & \dots & * & * & * & 1 & 1 & 1 \\ 0 & * & 0 & 1 & 1 & 0 & 1 & \dots & 1 & 0 & 1 & 1 & 0 & 1 \end{array}$$

(b) Type II

$$\begin{array}{cccccccc} 0 & * & * & * & * & * & * & \dots \\ * & \underline{0} & * & * & * & * & * & \dots \\ * & * & 0 & * & * & * & * & \dots \\ * & * & 1 & 1 & 1 & 1 & 1 & \dots \end{array}$$

(c) Type III

$$\begin{array}{cccccccccccc} 0 & * & * & * & * & * & * & * & * & * & * & \dots \\ * & \underline{0} & * & * & * & * & * & * & * & * & * & \dots \\ * & * & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \dots \end{array}$$

(d) Type IV

$$\begin{array}{ccccccccccc} \dots & * & * & * & * & \underline{1} & * & * & * & * & \dots \\ \dots & * & * & * & * & * & * & * & * & * & \dots \\ \dots & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & \dots \end{array}$$

(e) Type V

Forbidden constellations

$$\begin{array}{cccc|ccc|c|ccc|ccc} 0 & * & * & * & * & * & * & \dots & * & * & * & * & * & * \\ * & \underline{0} & * & * & * & * & * & \dots & * & * & * & 1 & 1 & 1 \\ * & * & 0 & 1 & 1 & 0 & 1 & \dots & 1 & 0 & 1 & 1 & 0 & 1 \end{array}$$

(a) Type I

$$\begin{array}{cccc|ccc|c|ccc|ccc} * & \underline{0} & * & * & * & * & * & \dots & * & * & * & 1 & 1 & 1 \\ 0 & * & 0 & 1 & 1 & 0 & 1 & \dots & 1 & 0 & 1 & 1 & 0 & 1 \end{array}$$

(b) Type II

$$\begin{array}{cccccccc} 0 & * & * & * & * & * & * & \dots \\ * & \underline{0} & * & * & * & * & * & \dots \\ * & * & 0 & * & * & * & * & \dots \\ * & * & 1 & 1 & 1 & 1 & 1 & \dots \end{array}$$

(c) Type III

$$\begin{array}{cccccccccccc} 0 & * & * & * & * & * & * & * & * & * & * & \dots \\ * & \underline{0} & * & * & * & * & * & * & * & * & * & \dots \\ * & * & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \dots \end{array}$$

(d) Type IV

$$\begin{array}{ccccccccccc} \dots & * & * & * & * & \underline{1} & * & * & * & * & \dots \\ \dots & * & * & * & * & * & * & * & * & * & \dots \\ \dots & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & \dots \end{array}$$

(e) Type V

... and additionally East↔West mirrored versions of Type I–IV

Riviera model (1D)