Shadowing doesn't imply $ICT = \omega(f)$

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Definitions

- X - compact, metric
- Dynamical system (X, f)

Definition (ω-limit set)

$$\omega(x) = \bigcap_{n \in \mathbb{N}} \overline{\{f^k(x) : k \geq n\}}$$

Definition (Weak incompressibility)

Closed non empty set $A \subset X$ that contains at least two different points is weakly incompressible, shorter WI, if for every relatively open subset U of A statement $\overline{f(U)} \cap A \subset U$ implies $U = \emptyset$ or $U = A$. If $A = \{a\}$ is singleton than we say that $\{a\}$ is WI only in case a being a fixed point for f.

Definitions

Theorem (Sarkovskii, 1965)

For every $x \in X$ set $\omega(x)$ is WI.

Definition (Internal chain transitivity)

Closed non empty set $A \subset X$ is internally chain transitive, shorter ICT, if for every $\delta > 0$ and every two points $a, b \in A$ there is δ-pseudo orbit from a to b.

Theorem (Good and Raines, [4])

WI=ICT

Example (Not every ICT is ω-limit set)

Doubling map (mod 1) on the set $\{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\}$. Whole set is ICT, but not ω-limit.

Shift spaces

- full shift $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ with map σ
- shift space - compact and σ-invariant subspace of Σ_2
- characterised via forbidden words
- shift of finite type - finite set of forbidden words

Theorem (2012, Barwell et al.)

In the shift of finite type, set is ICT iff it is ω-limit set of some point.

Shadowing

Definition (Shadowing)

System (X, f) has shadowing property if for every $\epsilon > 0$ there is $\delta > 0$ s.t. every δ-pseudo orbit is ϵ-shadowed.

Theorem (1978, Walters)

Shift space is of finite type iff it has shadowing.

Question

Does shadowing imply $ICT(f) = \omega(f)$?

Hausdorff hyperspace $(\mathcal{H}, d_{\mathcal{H}})$

- $\mathcal{H}$ - set of all compact subsets of X
- $d_{\mathcal{H}}(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}$
- metric $d_{\mathcal{H}}$ makes $\mathcal{H}$ a compact metric space
- $\omega(f)$ and $ICT(f)$ are subsets of $\mathcal{H}$ s.t. $\omega(f) \subset ICT(f)$

Meddaugh and Raines' results

Theorem (Meddaugh and Raines, 2013)

- *$ICT(f)$ is closed set in $\mathcal{H}$.*
- *If (X, f) has shadowing, then $\overline{\omega(f)} = ICT(f)$.*

Question (reformulation)

Does shadowing imply closedness of $\omega(f)$?

Example ($\omega(f)$ is not closed in general)

Take unit disc in complex plane and mapping $re^{i\phi} \mapsto re^{i(\phi+r)}$.
This system unfortunately hasn't shadowing.

Corollaries

Improving on a Blokh, Bruckner, Humke and Smital's result from 1996 for interval maps, Mai and Shao proved

Theorem (Mai and Shao, 2007)

Set $\omega(f)$ is closed for any dynamical system (G, f) given that G is graph.

Corollary (Meddaugh and Raines, 2013)

For graph maps shadowing implies $ICT(f) = \omega(f)$.

Example

- $X_k = \{\xi \in \Sigma_2 \mid$ any two 1's are separated with at least $k+1$ 0's$\}$
- $X_\infty = \{\xi \in \Sigma_2 \mid \xi$ has at most one 1$\}$
- Actually X_k is SFT with forbidden words $\{11, 101, \ldots, \underbrace{10\ldots01}_{k\text{-zeros}}\}$
- $N = \{\frac{1}{2^k} \mid k \in \mathbb{N} \cup \{0, \infty\}\}$ with convention $\frac{1}{2^\infty} = 0$
- X_∞ is conjugate to doubling map on $N \setminus \{1\}$ (sofic shift)

Theorem

Let

$$X = \{0\} \times X_\infty \cup \bigcup_{k=0}^{\infty} \{\frac{1}{2^k}\} \times X_k,$$

and $s(a, \xi) = (a, \sigma(\xi))$. System (X, s) has shadowing but set $\omega(s)$ isn't closed, consequently there is ICT set that isn't ω-limit set.

The End!

Thank you!

References

[1] Andrew D. Barwell, Gareth Davies, and Chris Good, *On the ω-limit sets of tent maps*, Fund. Math. **217** (2012), no. 1, 35–54.

[2] Peter Walters, *On the pseudo-orbit tracing property and its relationship to stability*, Lecture Notes in Math., vol. 668, Springer, Berlin, 1978, pp. 231–244.

[3] Jonathan Meddaugh and Brian E. Raines, *Shadowing and internal chain transitivity*.

[4] Andrew D. Barwell, Chris Good, Piotr Oprocha, and Brian E. Raines, *Characterizations of ω-limit sets in topologically hyperbolic systems*, Discrete Contin. Dyn. Syst. **33** (2013), no. 5, 1819–1833.

[5] Andrew Barwell, Chris Good, Robin Knight, and Brian E. Raines, *A characterization of ω-limit sets in shift spaces*, Ergodic Theory Dynam. Systems **30** (2010), no. 1, 21–31.

[6] Jie-Hua Mai and Song Shao, *Spaces of ω-limit sets of graph maps*, Fund. Math. **196** (2007), no. 1, 91–100.