Almost Totally Minimal Systems; Periodicity in Hyperspaces

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University of Birmingham

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Co-authors

Leobardo Fernández

Chris Good

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Almost minimal systems

Definition

Let X be a compact metric space and $T\colon X \to X$ a homeomorphism. We say that (X, T) is *almost minimal* if:

(1) There exists a unique fixed point $x_0 \in X$ s.t. $T(x_0) = x_0$
(2) The full orbit of every other point $y \in X \setminus \{x_0\}$ is dense

$$\overline{\{T^i(y) \mid i \in \mathbb{Z}\}} = X$$

N.B. $(X \setminus \{x_0\}, T|_{X\setminus\{x_0\}})$ is a well defined minimal non-compact system.

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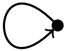
Do such systems exist?

The trivial example ✓



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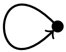
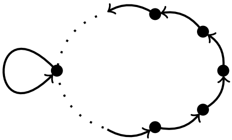


How about non-trivial?

$(\mathbb{Z} \cup \{\infty\}, +1)$ ✓



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Historical note

(1992) R. Herman, I. Putnam, C. Skau — relate K-theory and topological dynamics
(2001) A. Danilenko — extends their theory to non-compact setting by looking at *almost minimal* systems

ATMs

Definition

(X, T) is *almost totally minimal* if (X, T^k) is almost minimal for every $k \in \mathbb{N}$.

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X has to be perfect, and hence uncountable (why?)

X cannot be an interval (why?)

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How about the Cantor Set?

— Yes!

We use *graph covers* devised by:

(2006) J.-M. Gambaudo, M. Martens

(2008) E. Akin, E. Glassner, B. Weiss

(2014) T. Shimomura

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G_0 :

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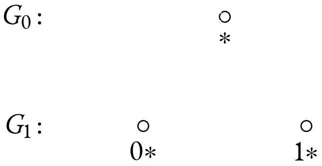

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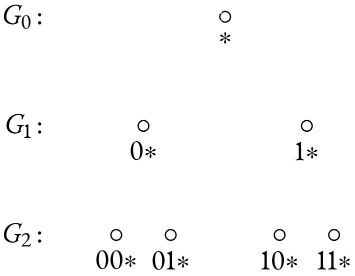

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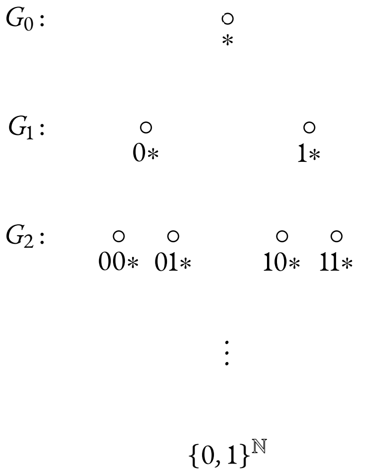

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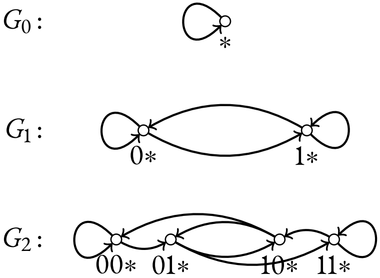
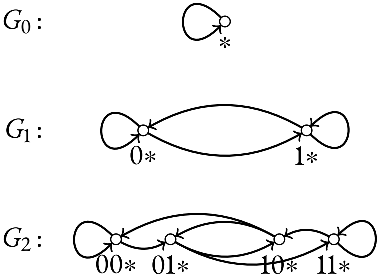


$$\vdots$$

$$(\{0,1\}^{\mathbb{N}}, \sigma)$$

— Yes!



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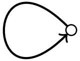
Compare

(1963) J. Mioduszewski

$(\{0,1\}^{\mathbb{N}}, \sigma)$

The construction

G_0 :



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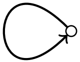
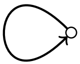


G_1 :

The construction

G_0 :



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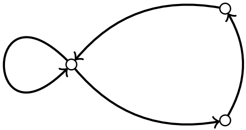

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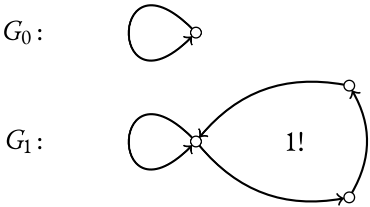

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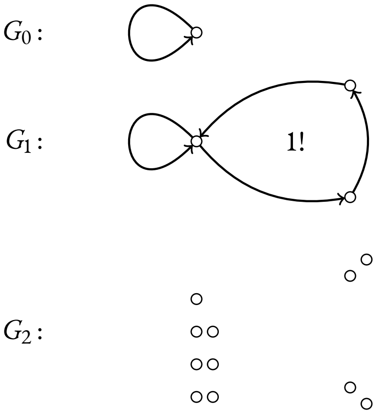

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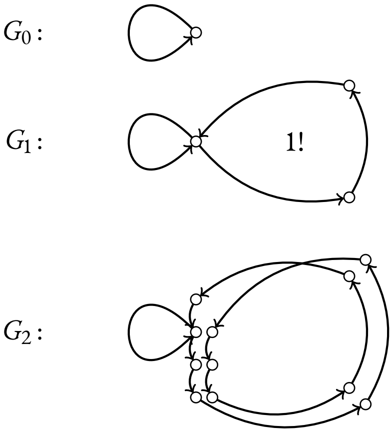

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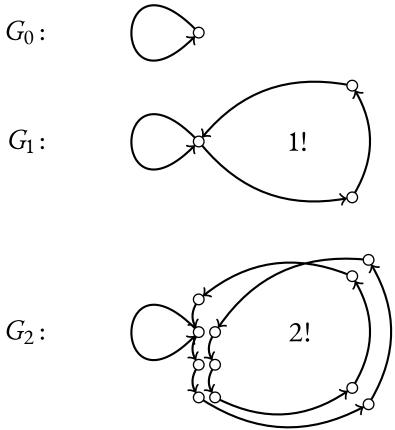

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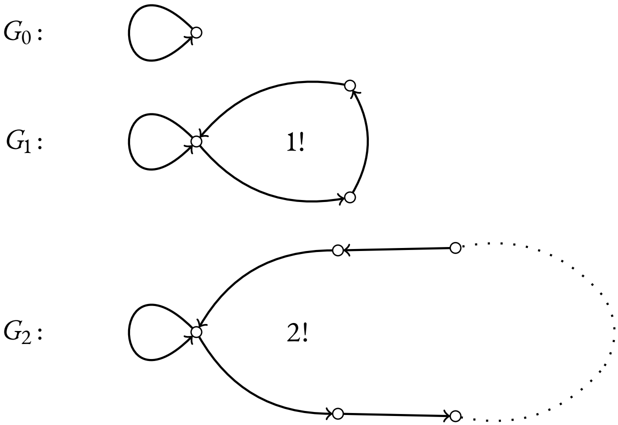

Any questions so far?

An Application

Theorem

Let (Y, S) be a 0-dimensional minimal system. There exist a system $(\hat{Y}, \hat{S})$ on the Cantor set $\hat{Y}$ such that:

(1) *(Y, S) dynamically embeds into $(\hat{Y}, \hat{S})$ as a nowhere dense set,*

(2) *Every full $\hat{S}^k$-orbit of any point $y \in \hat{Y} \setminus Y$ is dense in $\hat{Y}$ for every $k \in \mathbb{N}$.*

N.B. The nowhere density in (1) is redundant as it follows from (2).

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Proof.

(X, T) is the ATM constructed before with the fixed point x_0

$A \sqcup B = X$ is a separation s.t. x_0 is in A

Fix an arbitrary point $y_0 \in Y$ so that (x_0, y_0) acts as an origin

On $X \times Y$, consider $\hat{S} = \pi \circ (T \times S) \circ \pi$ where $\pi \colon (x, y) \mapsto \begin{cases} (x, y), & \text{if } x \in A \\ (x, y_0), & \text{if } x \in B \end{cases}$

Let $\hat{Y} \subset X \times Y$ be minimal (w.r.t. $\subseteq$) closed $\hat{S}$-invariant set containing $B \times \{y_0\}$

Then $(\hat{Y}, \hat{S}|_{\hat{Y}})$ works! □

A question remains

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What about the non-minimal case?
— It is true for some trivial non-minimal systems
(if they are a power of a minimal system)

Periodicity in Hyperspaces

We start with a problem

- X is compact metric
- $T\colon X \to X$ is continuous
- 2^X is the set of all compact subsets of X with the Hausdorff distance $d_{\mathrm{H}}(F, G) = \inf\{\varepsilon \geq 0 \mid F \subseteq G_\varepsilon \text{ and } G \subseteq F_\varepsilon\}$
- $2^T\colon 2^X \to 2^X$ is given by $2^T(F) = T(F)$
- $\mathrm{Per}(T) = \{k \in \mathbb{N} \mid \exists$ periodic point for T with fundamental period $k\}$
- $\mathrm{Per}(2^T)$ as above but for $(2^X, 2^T)$

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Šarkovs'kiĭ's theorem (1964)

For any interval map T:

$$3 \in \mathrm{Per}(T) \implies \mathrm{Per}(T) = \mathbb{N}.$$

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Characterise all admissible pairs $(\mathrm{Per}(T), \mathrm{Per}(2^T))$*?*

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Characterise all admissible pairs $(\mathrm{Per}(T), \mathrm{Per}(2^T))$*?*

Is there a system (X, T) *for which* $\mathrm{Per}(2^T) = \{1, 2, 3\}$*?*

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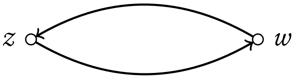

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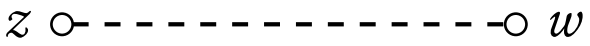

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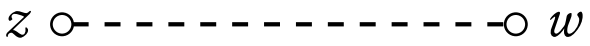

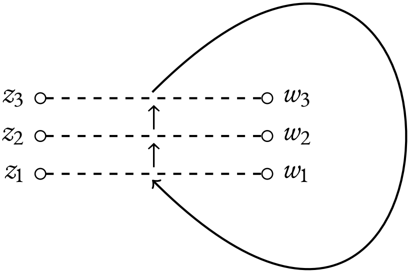

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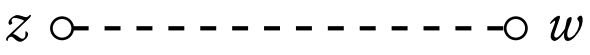

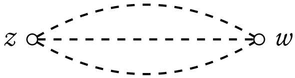

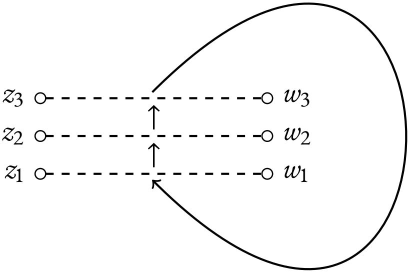

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Fact sheet I

Trivial periods coming from (X, f) can be characterised as

$$[\mathcal{D}(\mathrm{Per}(f))] = \bigcup_{l=1}^{\infty} \left\{ [d_1, \ldots, d_l] \mid d_i | m_i \in \mathrm{Per}(f) \text{ for } 1 \leq i \leq l \right\}$$

But there could be more!

Theorem

Given a continuous map $f : X \to X$, the set of periods $\mathrm{Per}(2^f)$ of the induced map on 2^X contains $[\mathcal{D}(\mathrm{Per}(f))]$ and is closed under taking prime power divisors.

Theorem

Let f be a continuous map of a compact interval to itself. Then $\mathrm{Per}(2^f)$ is either $\{1\}$ or $\{1, 2\}$ or $\mathbb{N}$.

Fact sheet II

Theorem (Shimomura)

Any surjective dynamics over a 0*-dimensional system is topologically conjugate to*

$$G_\infty = \varprojlim G_i = G_0 \xleftarrow{\phi_0} G_1 \xleftarrow{\phi_1} G_2 \xleftarrow{\phi_2} \cdots$$

where G_i are finite directed graphs (each vertex has at least one in- and out-edge), and bonding maps $\phi_i \colon G_{i+1} \to G_i$ are graph covers (vertex map that respects edges and +*-directional).*

A different proof I

$A \sqcup B = \mathscr{C}$, x^0 the fixed point of ATM $(\mathscr{C}, T)$ is in A
$\hat{X} = A \times \{0,1\} \sqcup B \times \{0\}$
$f : \hat{X} \to \hat{X}$ by

$$f(x,i) = \begin{cases} (T(x), 1), & \text{if } i = 0 \text{ and } x \in T^{-1}(A), \\ (T(x), 0), & \text{otherwise.} \end{cases}$$

$N(x, B) = \min\{k \in \mathbb{N}_0 \mid T^{-k}(x) \in B\}$ time elapsed since x last visited B
$U = \{x \in \mathscr{C} \mid N(x, B) < \infty\}$ dense and open
$X = \overline{\{(x, N(x, B) \bmod 2) \mid x \in U\}} \subset \hat{X}$ unique minimal closed f^m-invariant set containing $B \times \{0\}$ for any $m \in \mathbb{N}$

Lemma

For any $z \in X \setminus \pi^{-1}(x^0)$ and any $m \in \mathbb{N}$ we have

$$\overline{\{f^{mk}(z) \mid k \in \mathbb{Z}\}} = X$$

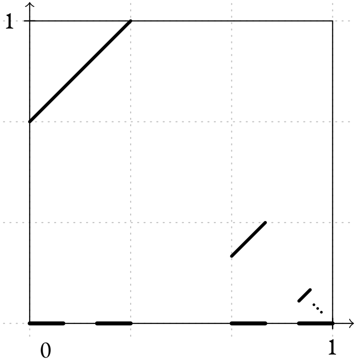
A different proof II

$l = (x^0, 0)$ and $r = (x^0, 1)$ form a 2-cycle in (X, f)
$Z = X \times \{0, 1, 2\}/_\sim$ a quotient space obtained by gluing $L = \{(l, i) \mid i = 0, 1, 2\}$ together and likewise $R = \{(r, i) \mid i = 0, 1, 2\}$
$g\colon Z \to Z, g(x, i) = (f(x), i + 1 \mod 3)$, well-defined
$\{L\} \mapsto \{R\}$ is a 2-cycle in 2^Z
$X \times \{0\} \mapsto X \times \{1\} \mapsto X \times \{2\}$ is a 3-cycle in 2^Z
No 6-cycle in 2^Z exists!
Otherwise let S be it; $\exists z = (z_1, i) \in S$ other than L or R
$\{g^{6k}(z) \mid k \in \mathbb{Z}\} \subset S$
$\overline{\{f^{6k}(z_1) \mid k \in \mathbb{Z}\}} \times \{i\} = X \times \{i\} \subset S$
Hence $S = X \times F/_\sim$ for some $F \subseteq \{0, 1, 2\}$ □

Example – 2-adic odometer

$T\colon \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$

$$T(x_1, x_2, \dots) = \begin{cases} (1, x_2, \dots), & \text{if } x_1 = 0, \\ (0, T(x_2, x_3, \dots)), & \text{o/w.} \end{cases}$$



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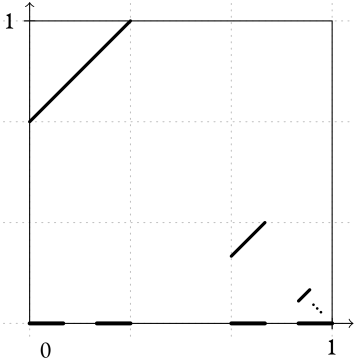
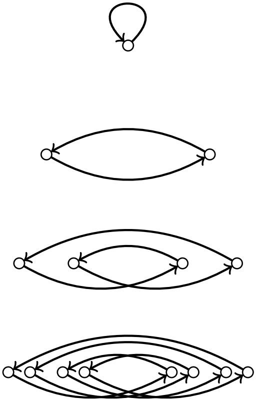


$G_0:$

$G_1:$

$G_2:$

$G_3:$



Bratteli-Vershik representation