~~Periodicity in hyperspaces; Almost totally minimal systems~~

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joint work w/ C. Good & L. Fernández $\rightsquigarrow$ published in ETDS

I Periodicity in hyperspaces

X cmpt metric

$f : X \to X$ conts map

(X, f) $\rightsquigarrow$ study iterations of f: $x, f(x), f^2(x), \dots$

2^X – hyperspace of cmpt, nonempty sets of X

$\hookrightarrow$ Also cmpt $\to$ Hausdorff distance

$$d_H(A,B) = \inf \{ \varepsilon \mid N(A,\varepsilon) \supseteq B \text{ and } N(B,\varepsilon) \supseteq A \}$$

2^f – induced map 2^X $\quad 2^f(S) = f(S) = \{ f(x) \mid x \in S \}$

2^f is conts map

$(2^X, 2^f)$ induced system

? What can be said dynamically about this induced system; periodic pts

$$\mathrm{Per}(f) = \{ n \in \mathbb{N} \mid \exists x \in X \text{ with fundamental period } n \}$$

$$\mathrm{Per}(2^f) = \{ n \in \mathbb{N} \mid \exists S \subseteq X \text{ cpt, nne with } f^n(S) = S \ \& \ f^k(S) \neq S \text{ for } 0<k<n \}$$

Observations: $\mathrm{Per}(2^f) \supseteq \mathrm{Per}(f)$

if $\mathrm{Per}(x) = n$ (in X) $\Rightarrow \mathrm{Per}(\{x\}) = n$ (in 2^X)

$1 \in \mathrm{Per}(2^f)$ $\qquad \mathrm{Per}(X) = 1$ (in 2^X)

Ex 1

$X = \{a,b,c,d,e\}$

$\mathrm{Per}(f) = \{2,3\}$

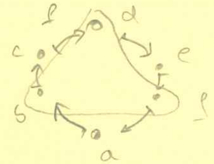
$\mathrm{Per}(2^f) = \{1,2,3,6\}$

$\mathrm{Per}(\{c,d\}) = 6$

More generally if $A \cap B = \emptyset$ $\quad \mathrm{Per}(A \cup B) = [\mathrm{Per}(A), \mathrm{Per}(B)]$ $\leftarrow$ the least common multiple

$$\mathrm{Per}(2^f) \supseteq [\mathrm{Per}(f)]$$

Ex 2 $\quad X = \{a,b,c,d,e,f\}$



$\mathrm{Per}(f) = \{6\}$

$\mathrm{Per}(2^f) = \{1, 2, 3, 6\}$ $\quad$ (2: $\{b,d,f\}$; 3: $\{a,d\}$)

$$\mathrm{Per}(2^f) \supseteq [\mathcal{D}(\mathrm{Per}(f))] = \mathcal{D}([\mathrm{Per}(f)])$$

closure of divisors

In a sense, this is the best we can do as for each $S \subseteq \mathbb{N}$ there is a system (X,f) with $\mathrm{Per}(f) = S$ and $\mathrm{Per}(2^f) = [\mathcal{D}(S)]$.

How? If S finite $\rightsquigarrow$ take a disjoint union of cycles, one for each $n \in S$

$\{ \cdot n_1 \cdot, \ \cdot n_2 \cdot, \dots \}$ done ✓

If S infinite $\rightsquigarrow$ same but add one fixed pt to compactify the space

This adds 1 to $\mathrm{Per}(f) = S$

$\rightsquigarrow$ if we wish to avoid this compactify with an irrational rotation

However, this still doesn't provide the full description of all subsets of $\mathbb{N}$ that can be $\mathrm{Per}(2^f)$.

Hw Is there (X,f) with $\mathrm{Per}(f) = \emptyset$, $\mathrm{Per}(2^f) = \mathbb{N}$

? Is there (X,f) st. $\mathrm{Per}(2^f) = \{1,2,3\}$? $\quad (*)$

Digression: What is the role of topology?

$\boxed{2}$

If we disregard the topology (*) cannot happen.
↳ allow all n-e subsets of X

Indeed, there are only 2 scenarios

$\boxed{1^{st}}$ $\exists x \in X$ with infinite bound orbit

$\dots \mapsto x_{-2} \mapsto x_{-1} \overset{f}{\mapsto} x_0 \mapsto x_1 \mapsto \dots$

$\forall k \in \mathbb{N}$ $\quad Per(\{\dots, x_{-k}, x_{-1}, x_0, x_k, x_{2k}, \dots\}) = k$ (in "2^X")

$Per(\text{"}2^X\text{"}) = \mathbb{N}$

$\boxed{2^{nd}}$ all pts are periodic in f [∞ fund orbit ⇒ ∞ bound orbit]

but then $Per(\text{"}2^X\text{"}) = [\,\mathcal{O}(Per(f))\,]$ (again {1,2,3} not possible)

no there's ($\boxed{3^{rd}}$ case) w (ℕ orbits) → not a problem if f surjective which we can assume WLOG)

Having allowed only compact sets one needs to take closure in $\underline{1^{st}}$ case which means that the 1st pt was dense orbit we get $Per(X) = 1$
⇝ ω- and α- limit sets play a role

VII Almost (totally) minimal systems

Def: An invertible system (X, f) is almost minimal if

(i) $\exists !\, x_0 \in X$ st $f(x_0) = x_0$ (unique fixed pt)

(ii) $\forall x \in X \setminus \{x_0\}$ $\quad \overline{\{f^k(x) \mid k \in \mathbb{Z}\}} = X$ (full orbit of every other pt is dense in X)

Ex 3 a) trivial b)

Def: An invertible system (X, f) is almost totally minimal if

(i) $\exists !\, x_0 \in X$ st $f(x_0) = x_0$

(ii) $(\forall x \in X \setminus \{x_0\})(\forall n \in \mathbb{N})$ $\quad \overline{\{f^{nk}(x) \mid k \in \mathbb{Z}\}} = X$

Examples? Not obvious that such a system (non-trivial) exists!

Thm 1 There exists a homeo of the Cantor set which is ATM!

Idea: Construction using inverse limit of directed finite graphs
(J.M. Gambaudo & M. Martens, E. Akin & E. Glasner & B. Weiss, T. Shimomura)

$X_0 \xleftarrow{\varphi_0} X_1 \xleftarrow{\varphi_1} X_2 \xleftarrow{\varphi_2} \dots$

f_i at each level can be a multivalued map but as long as, say, $\varphi_{i-1}(\underbrace{f_i(x)}_{\text{set of values}}) \in X_{i-1}$ is a well defined pt

↳ we obtain a well defined map
$f : \varprojlim(X_i, \varphi_i) \to \varprojlim(X_i, \varphi_i)$

Choosing length of big loops carefully and also φ_is one can control trajectories of f enough to obtain ATM. ∎

I like to think of Thm 1 as saying that there is this "wild" dynamics that "hides" one fixed pt.

[?] Can we "hide" more than a fixed pt?

Theorem 2 Let (Y,g) be a minimal 0-dim (= tot disc. cup metr) $\hookrightarrow (\forall y\in Y\ \omega_g(y)=\bigcap_j \overline{\bigcup_{k\geq j} g^{k}(y)}=Y)$

Then $\exists\ (\hat{Y},\hat{g})$ ~~extnd~~ $Y\hookrightarrow\hat{Y}$ a cont. embedding st.

(i) $\hat{g}|_Y = g$

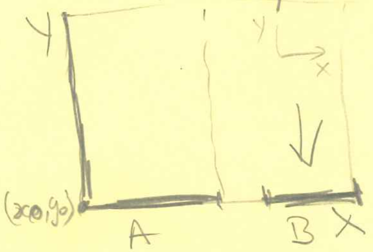
(ii) $Y\subseteq\hat{Y}$ is nowhere dense

(iii) $(\forall y\in\hat{Y}\setminus Y)(\forall n\in\mathbb{N})$ $\overline{\{\hat{g}^{nk}(y)\mid k\in\mathbb{Z}\}}=\hat{Y}$

Pf: ~~Let (X,f) be Note (iii) $\Rightarrow$ (ii)~~

Let (X,f) be ATM from Thm 1.

Take $(X\times Y,\ f\times g)$



Let $y_0\in Y$ be arbitrary

- Dynamics on $\{x_0\}\times Y$ is precisely g.
- But no mixing in Y-direction
- split X in two disjoint clopen sets $X = A\cup B$, $x_0\in A$
- let $\pi_B : X\times Y\to X\times Y$ be a projection

$$(x,y)\xmapsto{\pi_B}\begin{cases}(x,y) & \text{if } x\in A\\ (x,y_0) & \text{if } x\in B\end{cases}$$

Let $\hat{g}=\pi_B\circ(f\times g)$

$\hat{Y}=\overline{\bigcup_{k=0}^{\infty}\hat{g}^{k}(B\times\{y_0\})}$

We claim that this works!

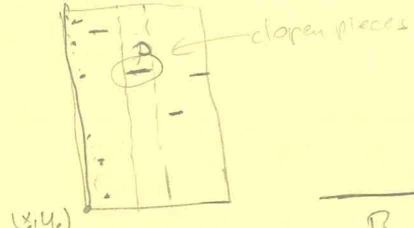
(i) Why is $\{x_0\}\times Y\in\hat{Y}$?

Because $\pi_X : \hat{X}\times Y\to X$ semiconjugates $(\hat{X},\hat{g})$ and (X,f)

$\Rightarrow \pi_X(\hat{X})=X \rightsquigarrow \exists\ y\in Y$ st. $(x_0,y)\in\hat{X} \overset{\text{minimality}}{\rightsquigarrow} \{x_0\}\times Y\subseteq\hat{Y}$

Clearly $\hat{g}|_{\{x_0\}\times Y}=g$ (i) ✓

(ii) follows from (iii)



$\bigcup_{k=0}^{\infty}\hat{g}^{k}(B\times\{y_0\})$ is open & dense in $\hat{Y}$

Now for (iii) note that

and π_X is injective on it

$\rightsquigarrow$ for each clopen piece P no other point from $\hat{X}$ intersects the strip $\pi_X^{-1}(\pi_X(P))$

We wish to show that $\forall (x,y)\in\hat{Y}$ with $x\neq x_0$ $\forall n\in\mathbb{N}$ $\overline{\{\hat{g}^{nk}(x,y)\mid k\in\mathbb{Z}\}}^{\wedge}=S$

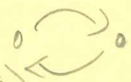
But by Thm 1 $\overline{\{f^{nk}(x)\mid k\in\mathbb{Z}\}}=X$ and so as before

$\pi_X(S)=X$ using semi-conjugacy argument.

But almost everywhere injectivity of π_X implies that $S=\hat{Y}$ ✓ (iii) ✓

Returning to $Per(f)=\{1,2,3\}$

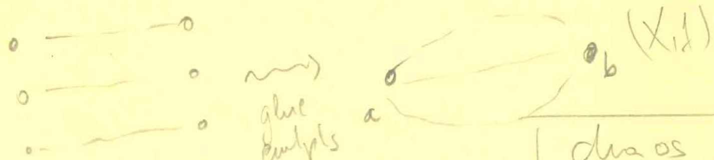
Take (Y,g) $\{a,b\}$



$\rightsquigarrow$ embed it as in Thm 2 $(\hat{Y},\hat{g})$

$\rightsquigarrow$ take product with 3-cycle

$(\hat{Y}\times\{0,1,2\},\ \hat{g}\times(\mathrm{add}_{1\bmod 3}))$



$\rightsquigarrow$ glue endpts a b (X,f)

chaos
- $LiYorke$ ✓ (weakly mixing)
- $h=0 \rightsquigarrow$ B/C ✗
- ~~Devany~~ ✗ (no periodic)

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Claim: $\mathrm{Per}(f) = \{2\}$

$\mathrm{Per}(2^f) = \{1,2,3\}$

Let $S \subseteq X$ (empty, n-e)

if $S \subseteq \{a,b\}$ then $\mathrm{Per}(S) = 1$ or 2

Else, $\exists\, x \in X \setminus \{a,b\} \cap S$. Assume that $\mathrm{Per}(S) = n$

But then $\overline{\bigcup_{k=0}^{\infty} f^{3nk}(x)} \subseteq S$

$\parallel$

the "whole line" in which x happens to be

$\rightsquigarrow$ analogously we can conclude $S =$ union of "some lines"

$\Rightarrow \mathrm{Per}(S) = 1$ or 3

$S =$ all 3 lines; $S =$ only 1 or 2

$\blacksquare$